

Periods and factors of weak model sets

Gerhard Keller and Christoph Richard *

Department Mathematik, Universität Erlangen-Nürnberg

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Abstract

There is a renewed interest in weak model sets due to their connection to \mathcal{B} -free systems [6], which emerged from Sarnak's program on the Möbius disjointness conjecture. Here we continue our recent investigation [15] of the dynamical system \mathcal{M}_W^G associated to a weak model set in a group G with window W . As a result, we identify the maximal equicontinuous factor of \mathcal{M}_W^G and give a sufficient condition when \mathcal{M}_W^G is an almost 1-1 extension of its maximal equicontinuous factor. If the window has positive Haar measure, the system \mathcal{M}_W^G equipped with its Mirsky measure is isomorphic to its Kronecker factor. For general nontrivial ergodic probability measures on \mathcal{M}_W^G , we provide a kind of lower bound for the Kronecker factor. All relevant factor systems are natural G -actions on quotient subgroups of the torus underlying the weak model set. These are obtained by factoring out suitable window periods. These results are also interpreted for \mathcal{B} -free systems.

1 Introduction

Fix two locally compact second countable abelian groups G and H . Typically, $G = \mathbb{Z}^d$ or \mathbb{R}^d , whereas H will often be a more general group. Each pair (\mathcal{L}, W) , consisting of a cocompact lattice $\mathcal{L} \subseteq G \times H$ and a relatively compact and measurable subset W of H , also called the window, defines a weak model set $\Lambda = \Lambda(\mathcal{L}, W)$ as the set of all points $x_G \in G$ for which there exists a point $x_H \in W$ such that $(x_G, x_H) \in \mathcal{L}$. Hence $\Lambda = \pi^G(\mathcal{L} \cap (G \times W))$. Such a set is sometimes also called a “cut-and-project set”. Model sets, which additionally satisfy $\text{int}(W) \neq \emptyset$, have been introduced by Meyer [18, 19]. There is an abundant literature on model sets, see e.g. the list of references in [2]. Weak model sets have been initially studied by Schreiber [23, 24]. The name weak model set was coined by Moody [21], see [10] for further background. An extended dynamical systems point of view on (weak) model sets was suggested in [15], which exploits the connection to the compact torus dynamical system $\hat{X} = (G \times H)/\mathcal{L}$ together with its canonical G -action. It first studies ergodic and topological properties of the associated G -action on an extension of the system one is really interested in - the benefit is that the results are most transparent for this extension - and then projects these properties to the true system of interest by a factor map π_*^G .

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Standard assumptions on the window are compactness, topological regularity and aperiodicity, i.e., $W = \overline{\text{int}(W)}$, and $h + W = W$ implies $h = 0$, see e.g. the discussion in [15, Sec. 4.3]. In that case, the map π_*^G is indeed a homeomorphism, so no information is lost. However relevant examples fall outside that class, even if one retains compactness of W . In this article, the nature of π_*^G is studied more systematically for compact windows. One of our results states that π_*^G is a homeomorphism whenever $\text{int}(W)$ is aperiodic. Hence for topological results the periods of $\text{int}(W)$ appear to play a central role. Indeed we obtain the maximal equicontinuous factor of \mathcal{M}_W^G by factoring from the group \hat{X} the periods of $\text{int}(W)$, see Theorem A2. For measure-theoretic results relating to the Mirsky measure Q_W^G , which is the natural lift of the Haar measure $m_{\hat{X}}$ on \hat{X} to \mathcal{M}_W^G , a central role is played by the so-called Haar periods of W , i.e., by those $h \in H$ satisfying $m_H((h + W) \Delta W) = 0$. Indeed the dynamical system (\mathcal{M}_W^G, Q_W^G) is isomorphic to its Kronecker factor, which is obtained from the torus \hat{X} by factoring out the Haar periods of W , compare Theorem B2. For any ergodic invariant probability measure P^G which is not supported on the zero configuration, we provide a kind of lower bound for the Kronecker factor of (\mathcal{M}_W^G, P^G) in Theorem C2. Weak model sets with a non-compact window may behave very differently. But the above structural results continue to hold for windows which are “close to compact” in a sense specified in Section 2.1(5). This setting encompasses many examples from the literature.

Our paper is organised as follows. After the setting has been explained in Section 2, we give formally precise statements of our results in Section 3. There we also discuss applications of the results to \mathcal{B} -free systems. Section 4 studies the question of reconstructing from a given weak model set a suitable window. This leads to a proof of Theorem A1, assuming that $\text{int}(W)$ is aperiodic. Proofs of measure theoretic statements for aperiodic windows are then provided in Section 5. As a preparation for the proofs of the remaining statements, period groups and quotient cut-and-project schemes are studied in Section 6. The following section contains the proofs of Theorems A2, B2 and C2. The final section discusses relatively compact windows whose associated dynamical systems behave very similarly to the ones with compact windows.

2 The setting

The following point of view on extended weak model sets was developed in [15].

2.1 Assumptions and notations

Certain spaces and mappings are needed for the construction of weak model sets. As in [15] we make the following general assumptions.

- (1) G and H are *locally compact second countable abelian groups* with Haar measures m_G and m_H . Then the product group $G \times H$ is locally compact second countable abelian as well, and we choose $m_{G \times H} = m_G \times m_H$ as Haar measure on $G \times H$.
- (2) $\mathcal{L} \subseteq G \times H$ is a *cocompact lattice*, i.e., a discrete subgroup whose quotient space $(G \times H)/\mathcal{L}$ is compact. Thus $(G \times H)/\mathcal{L}$ is a compact second countable abelian group. Denote by $\pi^G : G \times H \rightarrow G$ and $\pi^H : G \times H \rightarrow H$ the canonical projections. We assume that $\pi^G|_{\mathcal{L}}$ is 1-1 and that $\pi^H(\mathcal{L})$ is dense in H .¹

¹Denseness of $\pi^H(\mathcal{L})$ can be assumed without loss of generality by passing from H to the closure of $\pi^H(\mathcal{L})$. In that case m_H must be replaced by $m_{\overline{\pi^H(\mathcal{L})}}$.

- (3) G acts on $G \times H$ by translation: $T_g x = (g, 0) + x$.
- (4) Let $\hat{X} := (G \times H)/\mathcal{L}$. As we assumed that \hat{X} is compact, there is a measurable relatively compact fundamental domain $X \subseteq G \times H$ such that $x \mapsto x + \mathcal{L}$ is a bijection between X and \hat{X} . Elements of $G \times H$ (and hence also of X) are denoted as $x = (x_G, x_H)$, elements of \hat{X} as \hat{x} or as $x + \mathcal{L} = (x_G, x_H) + \mathcal{L}$, when a representative x of \hat{x} is to be stressed. We normalise the Haar measure $m_{\hat{X}}$ on \hat{X} such that $m_{\hat{X}}(\hat{X}) = 1$. Thus $m_{\hat{X}}$ is a probability measure.
- (5) The *window* W is a measurable relatively compact subset of H . For our topological dynamical results, we first assume that the window W is indeed compact and discuss extensions of the results to certain non-compact windows in Section 8. Our purely measure theoretic results are first stated and proved for compact windows as well, but they extend easily to windows which agree modulo Haar measure zero with a compact one. Some further measure theoretic results, which have an additional topological aspect, are only proved for compact windows.

2.2 Consequences of the assumptions

We list a few facts from topology and measure theory that follow from the above assumptions. We will call any neighborhood of the neutral element in an abelian topological group a *zero neighborhood*.

- (1) Being locally compact second countable abelian groups, G , H and $G \times H$ are metrisable with a translation invariant metric with respect to which they are complete metric spaces. In particular they have the Baire property. As such groups are σ -compact, m_G , m_H and $m_{G \times H}$ are σ -finite.
- (2) As $G \times H$ is σ -compact, the lattice $\mathcal{L} \subseteq G \times H$ is at most countable. Note that $G \times H$ can be partitioned by shifted copies of the relatively compact fundamental domain X . This means that \mathcal{L} has a positive finite point density $\text{dens}(\mathcal{L}) = 1/m_{G \times H}(X)$. We thus have $m_{\hat{X}}(\hat{A}) = \text{dens}(\mathcal{L}) \cdot m_{G \times H}(X \cap (\pi^{\hat{X}})^{-1}(\hat{A}))$ for any measurable $\hat{A} \subseteq \hat{X}$, where $\pi^{\hat{X}} : G \times H \rightarrow \hat{X}$ denotes the quotient map. As a factor map between topological groups, $\pi^{\hat{X}}$ is continuous and open.
- (3) The action $\hat{T}_g : \hat{x} \mapsto (g, 0) + \hat{x}$ of G on \hat{X} is minimal. This implies that \hat{X} with its natural action is uniquely ergodic, see e.g. [21, Prop. 1].
- (4) Denote by \mathcal{M} and \mathcal{M}^G the spaces of all locally finite measures on the Borel subsets of $G \times H$ and G , respectively. They are endowed with the topology of vague convergence and hence compact spaces. As G and $G \times H$ are complete metric spaces, this topology is Polish, see [12, Thm. A.2.3].

2.3 The objects of interest

The pair (\mathcal{L}, W) assigns to any point $\hat{x} \in \hat{X}$ a discrete point set in $G \times H$. We will identify such point sets P with the measure $\sum_{y \in P} \delta_y \in \mathcal{M}$ and call these objects *configurations*. More precisely:

- (1) For $\hat{x} = x + \mathcal{L} \in \hat{X}$ define the configuration

$$\nu_W(\hat{x}) := \sum_{y \in (x + \mathcal{L}) \cap (G \times W)} \delta_y. \quad (1)$$

It is important to understand ν_W as a map from \hat{X} to \mathcal{M} . If W is compact, the map ν_W is upper semicontinuous [15, Prop. 3.3]. The canonical projection $\pi^G : G \times H \rightarrow G$ projects measures $\nu \in \mathcal{M}$ to measures $\pi_*^G \nu$ on G defined by $\pi_*^G \nu(A) := \nu((\pi^G)^{-1}(A))$. We abbreviate

$$\nu_W^G := \pi_*^G \circ \nu_W : \hat{X} \rightarrow \mathcal{M}^G. \quad (2)$$

- (2) Denote by \mathcal{M}_w the vague closure of $\nu_w(\hat{X})$ in \mathcal{M} , and by \mathcal{M}_w^G the vague closure of $\nu_w^G(\hat{X})$ in \mathcal{M}^G . The group G acts continuously by translations on all these spaces: $(S_g \nu)(A) := \nu(T_g^{-1} A) = \nu(T_{-g} A)$. Here we used the same notation S_g for translations on \mathcal{M}_w and \mathcal{M}_w^G , as the meaning will always be clear from the context.
- (3) As $\nu_w(\hat{x})(T_{-g} A) = (S_g \nu_w(\hat{x}))(A) = \nu_w(\hat{T}_g \hat{x})(A)$, it is obvious that all $\nu_w(\hat{x})$ are uniformly translation bounded, and it follows from [4, Thm. 2] that all four spaces from item (2) are compact.
- (4) $Q_w := m_{\hat{X}} \circ \nu_w^{-1}$ and $Q_w^G := m_{\hat{X}} \circ (\nu_w^G)^{-1}$ are called Mirsky measures on \mathcal{M}_w and \mathcal{M}_w^G , respectively. Note that $Q_w^G = Q_w \circ (\pi_*^G)^{-1}$.²

2.4 Previous results

For compact windows, Mirsky measures on \mathcal{M}_w or on \mathcal{M}_w^G were studied in quite some detail in [15]. The following property is immediate from measurability of the map $\nu_w : \hat{X} \rightarrow \mathcal{M}_w$ and from the definition of the Mirsky measure Q_w on \mathcal{M}_w .

Proposition 2.1. (\mathcal{M}_w, Q_w, S) is a measure theoretic factor of $(\hat{X}, m_{\hat{X}}, \hat{T})$. □

In this article, we aim at statements concerning general invariant probability measures on \mathcal{M}_w or on \mathcal{M}_w^G . This is achieved using a partial inverse of ν_w : Denote by $\underline{0} \in \mathcal{M}$ the zero measure (“empty configuration”). We have $\underline{0} \in \mathcal{M}_w$ if and only if $\text{int}(W) = \emptyset$ by [15, Prop. 3.3]. Recall from [15, Lem. 5.4] that, for each $\nu \in \mathcal{M}_w \setminus \{\underline{0}\}$, there is a unique $\hat{\pi}(\nu) \in \hat{X}$, its “torus parameter”, such that $\text{supp}(\nu) \subseteq \text{supp}(\nu_w(\hat{\pi}(\nu)))$. This yields a continuous map $\hat{\pi} : \mathcal{M}_w \setminus \{\underline{0}\} \rightarrow \hat{X}$, and we have $\hat{\pi} \circ \nu_w = \text{id}$ whenever this composition is well defined, compare [15, Lem. 5.6].

The following observation is a measure theoretic analogue to Theorem 1a in [15]. Its proof, which is already implicit in the proof of [15, Thm. 2a], will be given in Subsection 8.2.

Proposition 2.2. Assume that P is any S -invariant probability measure on \mathcal{M}_w satisfying $P(\mathcal{M}_w \setminus \{\underline{0}\}) = 1$. Then $m_H(W) > 0$, and (\mathcal{M}_w, P, S) is a measure theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$.

Specialising to the Mirsky measure, we can combine the above two propositions and recover the following result. For the convenience of the reader, its proof will be given in Subsection 8.2.

Proposition 2.3 (Theorem 2a in [15]). Assume that $m_H(W) > 0$. Then (\mathcal{M}_w, Q_w, S) is measure theoretically isomorphic to $(\hat{X}, m_{\hat{X}}, \hat{T})$.

The projection $\pi^G : G \times H \rightarrow G$ induces a continuous factor map $\pi_*^G : (\mathcal{M}_w, S) \rightarrow (\mathcal{M}_w^G, S)$, which is the object of interest in this article. In order to understand to which extent statements as in the above propositions carry over to the system (\mathcal{M}_w^G, S) , one has to understand the degree of (non)invertibility of π_*^G .

Recall that a window $W \subseteq H$ is (topologically) aperiodic or irredundant, if $h + W = W$ implies $h = 0$. In particular, any aperiodic window is nonempty. A window $W \subseteq H$ is called topologically regular if $W = \overline{\text{int}(W)}$. Note that if $W \neq \emptyset$ is topologically regular, then $m_H(W) > 0$. The following fact is the essence of the results in [15, Sec. 4.3]:

Fact A. $\pi_*^G : \mathcal{M}_w \rightarrow \mathcal{M}_w^G$ is a homeomorphism, whenever the window W is aperiodic and topologically regular.

²These measures were denoted Q_M resp. Q_{M^G} in [15].

We now turn to measures on \mathcal{M}_W^G . For the Mirsky measure Q_W^G on \mathcal{M}_W^G , Proposition 2.1 immediately implies that $(\mathcal{M}_W^G, Q_W^G, S)$ is a measure theoretic factor of $(\hat{X}, m_{\hat{X}}, \hat{T})$, but there is no statement analogous to Proposition 2.2. From Fact A and Proposition 2.3, however, we get the following result for the Mirsky measure Q_W^G on \mathcal{M}_W^G .

Fact B. *Suppose that W is aperiodic and topologically regular. Then $(\mathcal{M}_W^G, Q_W^G, S)$ is measure theoretically isomorphic to $(\hat{X}, m_{\hat{X}}, \hat{T})$.* \square

The following is an immediate consequence of Fact A and Proposition 2.2.

Fact C. *Suppose that W is aperiodic and topologically regular. If P^G is an S -invariant probability measure on \mathcal{M}_W^G , then $(\mathcal{M}_W^G, P^G, S)$ is a measure theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$.* \square

In this article, we will extend the statements in the above three facts.

3 Main results

3.1 Topological results

In this subsection we assume that the window W is compact. Extensions of the results to certain non-compact windows are discussed in Section 8. Our first main result strengthens Fact A considerably.

Theorem A1. *Assume that $\text{int}(W)$ is aperiodic (so in particular non-empty).*

- a) *The topological dynamical systems (\mathcal{M}_W, S) and (\mathcal{M}_W^G, S) are isomorphic, and both are almost 1-1 extensions of their maximal equicontinuous factor (\hat{X}, \hat{T}) .*
- b) *Denote by $\Gamma : \mathcal{M}_W^G \rightarrow \hat{X}$ the factor map from a). If M is a non-empty, closed S -invariant subset of \mathcal{M}_W^G , then (M, S) is an almost 1-1 extension of its maximal equicontinuous factor (\hat{X}, \hat{T}) with factor map $\Gamma|_M$.*

- Remark 3.1.**
- a) The above result extends [15, Cor. 1], as aperiodicity of $\text{int}(W)$ and of $\overline{\text{int}(W)}$ are equivalent by Lemma 6.1. In particular, the result applies to $M = \mathcal{M}_{\min}^G$ the unique minimal subset of \mathcal{M}_W^G , see Remark 4.2, and to $M = \mathcal{M}_W^G(\hat{x}) := \{S_g \nu_W^G(\hat{x}) : g \in G\}$ the so-called *hull* of $\nu_W^G(\hat{x}) \in \mathcal{M}_W^G$.
 - b) Recall that $\text{int}(W) \neq \emptyset$ is aperiodic whenever the only compact subgroup of H is the trivial one, compare the proof of [15, Prop. 4.8]. This holds in particular for $H = \mathbb{R}^d$.
 - c) Aperiodicity of $\text{int}(W)$ implies aperiodicity of W . Examples of aperiodic W with periodic interior are presented in [13].
 - d) For a topologically regular window, aperiodicity of W and $\text{int}(W)$ are equivalent by Lemma 6.1. This gives the setting of an earlier result of Robinson [22, Th. 5.19, Cor. 5.20], see also [9, Prop. 7.3] in the Euclidean situation.

If $\text{int}(W)$ has non-trivial periods, we still can determine the maximal equicontinuous factor of (\mathcal{M}_W^G, S) . Given a subset $A \subseteq H$, we call

$$H_A := \{h \in H : h + A = A\}$$

the *period group* of A . The set $A \subseteq H$ is (topologically) *aperiodic*, if $H_A = \{0\}$. The following result extends Theorem A1. We denote by $\mathcal{H}_A := \{0\} \times H_A \subseteq G \times H$ the canonical embedding of H_A into $G \times H$. Note that $\mathcal{H}_{\text{int}(W)}$ is compact as $H_{\text{int}(W)} = \overline{H_{\text{int}(W)}}$ is a compact subgroup of H , compare Lemma 6.1.

Theorem A2. Assume that $\text{int}(W) \neq \emptyset$. Let $\widehat{X}' = \widehat{X}/\pi^\times(\mathcal{H}_{\text{int}(W)})$ with induced G -action \widehat{T}' , and let M be any non-empty, closed S -invariant subset of \mathcal{M}_W^G (thus including the case $M = \mathcal{M}_W^G$).

a) $(\widehat{X}', \widehat{T}')$ is the maximal equicontinuous factor of the topological dynamical system (M, S) .

b) If $H_{\text{int}(W)} = H_W$, then (M, S) is an almost 1-1 extension of $(\widehat{X}', \widehat{T}')$.

Remark 3.2. a) We do not know whether $H_{\text{int}(W)} = H_W$ is also a necessary condition in part b) of the theorem.

b) If $\text{int}(W) = \emptyset$, then the maximal equicontinuous factor of (M, S) is trivial, see Remark 4.2. However statements analogous to the above two theorems hold for maximal equicontinuous generic factors [14].

3.2 Measure theoretic results

In this subsection we assume that $W \subseteq H$ is relatively compact and measurable, but not necessarily compact. If $\text{int}(W) = \emptyset$, one should not expect that π_*^G is a homeomorphism. However, the measure theoretic statements of Facts B and C can still be generalized substantially to windows where $\pi_*^G : \mathcal{M}_W \rightarrow \mathcal{M}_W^G$ is not necessarily a homeomorphism, but still 1-1 on a sufficiently large subset of \mathcal{M}_W . This is achieved by replacing topological aperiodicity through a stronger measure theoretic version. In the following definition, Δ denotes the symmetric set difference.

Definition 3.3 (Haar aperiodicity). A measurable set $A \subseteq H$ is Haar aperiodic, if $m_H((h + A)\Delta A) = 0$ implies $h = 0$.

Remark 3.4. Any Haar aperiodic set A satisfies $m_H(A) > 0$ and is, in particular, nonempty. Haar aperiodicity implies topological aperiodicity, but the converse holds only for a more restricted class of windows, see Remark 3.12.

Definition 3.5. A measurable subset A of H is compact modulo 0, if there is a compact set $K \subseteq H$ such that $m_H(A\Delta K) = 0$.

We have the following results for the Mirsky measure Q_W^G on \mathcal{M}_W^G , which generalise Fact B. They are proved in Section 5.

Theorem B1. Suppose that W is compact modulo 0 and Haar aperiodic. Then $(\mathcal{M}_W^G, Q_W^G, S)$ is measure theoretically isomorphic to $(\widehat{X}, m_{\widehat{X}}, \widehat{T})$.

Remark 3.6. For a corresponding result about hulls $\mathcal{M}_W^G(\hat{x}) \subseteq \mathcal{M}_W^G$, assume that $\nu_W^G(\hat{x})$ has maximal density, compare Remark 8.7. In that case $Q_W^G(\mathcal{M}_W^G(\hat{x})) = 1$, see [15, Cor. 5], and the two systems $(\mathcal{M}_W^G(\hat{x}), Q_W^G, S)$ and $(\mathcal{M}_W^G, Q_W^G, S)$ are measure theoretically isomorphic.

Here is an extension of Theorem B1 to windows that are not Haar aperiodic. The proof is provided in Section 7 after some preparations in Section 6. To formulate the statement, we consider the group H_W^{Haar} of Haar periods of W , i.e.,

$$H_W^{\text{Haar}} := \{h \in H : m_H((h + W)\Delta W) = 0\}.$$

We write $\mathcal{H}_W^{\text{Haar}} = \{0\} \times H_W^{\text{Haar}}$ for the canonical embedding of H_W^{Haar} into $G \times H$.

Theorem B2. Suppose that W is compact modulo 0 and $m_H(W) > 0$. Let $\widehat{X}' = \widehat{X}/\pi^\times(\mathcal{H}_W^{\text{Haar}})$ with induced G -action \widehat{T}' and Haar measure $m_{\widehat{X}'}$. Then the measure theoretic dynamical system $(\mathcal{M}_W^G, Q_W^G, S)$ is isomorphic to $(\widehat{X}', m_{\widehat{X}'}, \widehat{T}')$.

The two final results refer to general invariant measures on \mathcal{M}_w^G and generalise Fact C. They are of *measure theoretic* nature, but they provide information about arbitrary ergodic measures on \mathcal{M}_w^G - the *topological* closure of $\nu_w(\hat{X})$. Therefore they are stated and proved only for compact windows. Hence in the remainder of this subsection we assume that $W \subseteq H$ is compact.

Before we can state the results, we need to introduce one more concept. For each $\nu \in \mathcal{M}_w$, $\pi_*^H \nu$ is a measure³ on H , and we denote the topological support of this measure by $\mathcal{S}_H(\nu)$. We thus have

$$\mathcal{S}_H(\nu) = \text{supp}(\pi_*^H(\nu)) = \overline{\pi^H(\text{supp}(\nu))} \subseteq W. \quad (3)$$

Remark 3.7. a) Clearly $\mathcal{S}_H(\underline{0}) = \emptyset$. Recall that there is a unique $\hat{x} \in \hat{X}$ such that $\nu \leq \nu_{\overline{W}}(\hat{x})$ if $\nu \neq \underline{0}$.

The set $\mathcal{S}_H(\nu)$ is the smallest compact set $W' \subseteq W$ such that $\nu \leq \nu_{W'}(\hat{x})$.

- b) We have $\overline{\text{int}(W)} \subseteq \mathcal{S}_H(\nu) \subseteq W$ for any $\nu \in \mathcal{M}_w$ by Lemma 4.3. The lower bound is attained for any continuity point $\hat{x} = x + \mathcal{L}$ of the map ν_w , because $\mathcal{S}_H(\nu_w(\hat{x})) = \overline{\text{int}(W) \cap \pi^H(x + \mathcal{L})} = \overline{\text{int}(W)}$ for such \hat{x} by Eqn. (6) below.
- c) In Lemma 4.5 we prove: For each ergodic S -invariant probability measure P on \mathcal{M}_w , there is a compact subset $W_P \subseteq W$ of H such that $\mathcal{S}_H(\nu) = W_P$ for P -a.a. ν . It should be no surprise that $m_H(W \Delta W_{Q_W}) = 0$ when W is compact modulo 0, see Corollary 5.2.

The first result is a consequence of Proposition 5.5 below.

Theorem C1. Suppose that W is compact and $m_H(W) > 0$. Let P^G be an ergodic S -invariant probability measure on \mathcal{M}_w^G , and let P be any ergodic S -invariant probability measure on \mathcal{M}_w satisfying $P^G = P \circ (\pi_*^G)^{-1}$. (There always exists at least one such measure P , see Proposition 5.5.) Suppose that W_P is aperiodic. (In particular $W_P \neq \emptyset$, i.e. $P(\mathcal{M}_w \setminus \{\underline{0}\}) = P^G(\mathcal{M}_w^G \setminus \{\underline{0}\}) = 1$.) Then π_*^G is a measure theoretic isomorphism between (\mathcal{M}_w, P, S) and $(\mathcal{M}_w^G, P^G, S)$, and $(\mathcal{M}_w^G, P^G, S)$ is a measure theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$.

Remark 3.8. a) As W_{Q_W} is aperiodic iff W is Haar aperiodic by Remark 3.12, the above statement is consistent with Theorem B1. Aperiodicity of W_P holds in \mathbb{R}^d , see Remark 3.1.

- b) The above result does not depend on the choice of P . Indeed we have $H_{W_P} = H_{W_{P'}}$, whenever P and P' are ergodic S -invariant measures on \mathcal{M}_w with $P \circ (\pi_*^G)^{-1} = P' \circ (\pi_*^G)^{-1}$. This holds as there is $d \in H$ such that $W_P = W_{P'} + d$ by Lemma 4.5. Hence, if $h \in H_{W_{P'}}$, then $h + W_P = h + (W_{P'} + d) = (h + W_{P'}) + d = W_{P'} + d = W_P$, so that $H_{W_{P'}} \subseteq H_{W_P}$, and the reverse inclusion follows from interchanging the roles of P and P' .

Again there is a periodic generalisation of this theorem, which is proved in Section 7.

Theorem C2. Suppose that W is compact and $m_H(W) > 0$. Let P^G be an ergodic S -invariant probability measure on $\mathcal{M}_w^G \setminus \{\underline{0}\}$. Take any ergodic S -invariant probability measure P on \mathcal{M}_w satisfying $P^G = P \circ (\pi_*^G)^{-1}$. (There always exists at least one such measure P , see Proposition 5.5.) Let $\hat{X}' = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{W_P})$ with induced G -action \hat{T}' and Haar measure $m_{\hat{X}'}$. Then $(\mathcal{M}_w^G, P^G, S)$ is a measure theoretic extension of $(\hat{X}', m_{\hat{X}'}, \hat{T}')$.

Remark 3.9. The above result does not depend on the choice of P , see Remark 3.8b).

For some of the proofs and applications, the following notion of Haar regularity appears to be relevant, compare Lemma 4.5. It is a measure theoretic substitute for topological regularity.

³ Note that generally $\pi_*^H \nu$ is not a Borel measure as ν may be an unbounded configuration and the topological support of $\pi_*^H \nu$ lies inside the compact set W .

Definition 3.10 (Haar regularity). *Consider some compact subset K of H .*

- a) *The compact set K_{reg} , defined as the topological support of the measure $(m_H)|_K$, is called the Haar regularization of K .*
- b) *The set K is Haar regular, if $K = K_{reg}$.*

Remark 3.11. a) A compact set $K \subseteq H$ is Haar regular if and only if for every open $U \subseteq H$ such that $U \cap K \neq \emptyset$ we have $m_H(U \cap K) > 0$.

b) $\text{int}(\overline{K}) \subseteq K_{reg} \subseteq K$ and $m_H(K \setminus K_{reg}) = 0$ by definition.

c) The empty set is Haar regular. If $K \neq \emptyset$ is Haar regular, then $m_H(K) > 0$.

d) If K is Haar regular, then any translate of K is Haar regular. If K is topologically regular, then K is Haar regular.

Remark 3.12. (Periods and Haar periods)

A compact set W is Haar aperiodic if and only if W_{reg} is aperiodic. More generally, $H_W^{Haar} = H_{W_{reg}}$. Indeed, as $W_{reg} \subseteq W$, we have $h + W_{reg} = W_{reg}$ if and only if $m_H((h + W) \Delta W) = 0$, which implies the claimed equivalence. In particular, every aperiodic Haar regular window is Haar aperiodic. Examples are Haar regular windows in \mathbb{R}^d or the windows defining taut \mathcal{B} -free systems, see Subsection 3.3.

Remark 3.13. (Haar regularity and compactness modulo 0)

If $A \subseteq H$ is compact modulo 0, then there is a compact set $K \subseteq H$ such that $m_H(A \Delta K) = 0$. Hence also $m_H(A \Delta K_{reg}) = 0$, and K_{reg} is the unique Haar regular set with this property. Therefore we denote it by A_{reg} . Observe that $A_{reg} = \text{supp}((m_H)|_A)$.

3.3 Applications to \mathcal{B} -free dynamics

General \mathcal{B} -free dynamical systems were studied in [6]. They are a special case of our systems (\mathcal{M}_w^G, S) , when $G = \mathbb{Z}$ and H is a particular compact group constructed from the given set $\mathcal{B} \subseteq \mathbb{N}$. In this setting a configuration $\nu^G = \sum_{n \in P} \delta_n \in \mathcal{M}$, where P is a subset of $G = \mathbb{Z}$, can be identified with the characteristic function χ_P interpreted as an element of $\{0, 1\}^{\mathbb{Z}}$. Our Theorem B1 reproduces Theorem F of [6] in this context, *i.e. the measure theoretic dynamical systems $(\mathcal{M}_w^G, \mathcal{Q}_w^G, S)$ and $(\hat{X}, m_{\hat{X}}, \hat{T})$ are isomorphic.*

Since the connection with [6, Theorem F] is not completely obvious, we give some explanation: We assume that the set \mathcal{B} is *primitive*, i.e. that no number from \mathcal{B} is a multiple of another number from \mathcal{B} . The following group homomorphism is associated with the set \mathcal{B} :

$$\Delta_{\mathcal{B}} : \mathbb{Z} \rightarrow \prod_{b \in \mathcal{B}} \mathbb{Z}/b\mathbb{Z}, \quad \Delta_{\mathcal{B}}(n) = (n \bmod b)_{b \in \mathcal{B}},$$

and H is the topological closure of $\Delta_{\mathcal{B}}(\mathbb{Z})$. The lattice is $\mathcal{L} = \{(n, \Delta_{\mathcal{B}}(n)) \in \mathbb{Z} \times H : n \in \mathbb{Z}\}$, and a moment's reflection shows that the group $\hat{X} = (\mathbb{Z} \times H)/\mathcal{L}$ is isomorphic to H . The window is defined as

$$W = \{h \in H : h_b \neq 0 \ \forall b \in \mathcal{B}\}. \quad (4)$$

The authors of [6] study, in our notation, the dynamics of S restricted to $\overline{\nu_w(\Delta_{\mathcal{B}}(\mathbb{Z}))}$, and denote this set by X_{η} . Our $\nu_w^G : \hat{X} \rightarrow \mathcal{M}_w^G$ corresponds to their $\varphi : H \rightarrow \{0, 1\}^{\mathbb{Z}}$ [6, Section 2.7]⁴, and so their definition of a Mirsky measure ν_{η} [6, Section 2.9] translates precisely to our \mathcal{Q}_w^G .

Finally observe that the tautness assumption on \mathcal{B} in [6, Thm. F] is equivalent to the Haar regularity of the window W defined in Eqn. (4), see [13, Thm. A], and that W is Haar aperiodic by Remark 3.12, because it is Haar regular and aperiodic [13, Prop. 5.1].

⁴Observe that the compact group H is denoted by G in [6].

In view of the preceding discussion, also our Theorem C1 applies to \mathcal{B} -free systems. It complements Theorem I from [6], which we recall here using our notation:⁵

For any ergodic S -invariant probability measure P^G on X_η , there exists a S -invariant probability measure ρ on $X_\eta \times \{0, 1\}^\mathbb{Z}$ whose first marginal is Q_W^G and such that $\rho \circ M^{-1} = P^G$, where $M : X_\eta \times \{0, 1\}^\mathbb{Z} \rightarrow X_\eta$ stands for the coordinatewise multiplication.

Together with our Theorem C1, which adds the lower arrow, this yields the following commutative diagram for measures P^G with an aperiodic associated window W_P :

$$\begin{array}{ccc}
 & (X_\eta \times \{0, 1\}^\mathbb{Z}, \rho, S \times S) & \\
 \pi_{X_\eta} \swarrow & & \searrow M \\
 (X_\eta, Q_W^G, S) & \xleftarrow{\nu_W^G \circ \hat{\pi} \circ (\pi_S^G)^{-1}} & (X_\eta, P^G, S)
 \end{array} \tag{5}$$

In [6, Theorem 8.2] the authors prove that the system (X_η, S) has a unique invariant measure $P_{max}^G = \rho_{max} \circ M^{-1}$ of maximal entropy, whenever the set \mathcal{B} has light tails⁶ and contains an infinite pairwise coprime subset. In fact, $\rho_{max} = Q_W^G \times B(1/2, 1/2)$, where $B(1/2, 1/2)$ denotes the Bernoulli $(1/2, 1/2)$ measure on $\{0, 1\}^\mathbb{Z}$. Here we complement their result by noticing that diagram (5) applies to P_{max}^G :

Proposition 3.14. *If \mathcal{B} has light tails and contains an infinite pairwise coprime subset, then the dynamical system (X_η, P_{max}, S) is a measure theoretic extension of $(\hat{X}, m_{\hat{X}}, \hat{T})$.*

Proof. We must show that the window $W_{P_{max}}$ is aperiodic: The entropy $h(P_{max}^G)$ coincides with the topological entropy of the system, which in turn equals the asymptotic density $d(\mathcal{F}_{\mathcal{B}})$ of the set $\mathcal{F}_{\mathcal{B}}$ of \mathcal{B} -free numbers, when the logarithm to base 2 is used to compute the entropy, see [6, Theorem D and Proposition K]. Remark 4.2 of [6] then shows that

$$\begin{aligned}
 h(P_{max}^G) &= d(\mathcal{F}_{\mathcal{B}}) = Q_W^G \{ \nu \in \mathcal{M}_W^G : \nu(\{0\}) > 0 \} = m_{\hat{X}} \{ \hat{x} \in \hat{X} : \nu_W^G(\hat{x})(\{0\}) > 0 \} \\
 &= m_H \{ h \in H : \varphi(h)(0) = 1 \} = m_H(W) .
 \end{aligned}$$

Fix any ergodic invariant measure P_{max} on \mathcal{M}_W that projects to P_{max}^G (compare Proposition 5.5a). This measure is supported by the set $\hat{\mathcal{M}}_{W'} := \{ \nu \in \mathcal{M}_W : \exists \hat{x} \in \hat{X} \text{ s.t. } \nu \leq \nu_{W'}(\hat{x}) \}$, where the window W' is defined as $W' = W_{P_{max}} \subseteq W$, and so P_{max}^G is supported by $\hat{\mathcal{M}}_{W'}^G := \{ \nu^G \in \mathcal{M}_W^G : \exists \hat{x} \in \hat{X} \text{ s.t. } \nu^G \leq \nu_{W'}^G(\hat{x}) \}$.

Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that $\text{card}\{k \in \mathbb{N} : |k| \leq n, \nu^G\{k\} = 1\} \leq n \cdot (m_H(W') + \epsilon)$ for all $n \geq N$ and all $\nu^G \in \hat{\mathcal{M}}_{W'}^G$, see [20, Theorem 1] or [15, Thm. 3]. Hence the same holds for all $\nu^G \in \hat{\mathcal{M}}_{W'}^G$. Therefore the topological entropy of the subshift $\hat{\mathcal{M}}_{W'}^G \subseteq \{0, 1\}^\mathbb{Z}$ is at most $m_H(W') + \epsilon$. It follows that $m_H(W) = h(P_{max}^G) \leq m_H(W')$. (For related arguments see also [10].) Hence $m_H(W \setminus W') = 0$. Because sets \mathcal{B} with light tails are taut [6, Section 2.6] and give rise to Haar regular windows [13, Theorem A], this implies $W_{P_{max}} = W' = W$. As, in the context of \mathcal{B} -free systems, the window W is always aperiodic [13], we see that $W_{P_{max}}$ is aperiodic. \square

⁵To be precise, [6, Thm. I] is stated for general, not necessarily ergodic measures P^G , and it applies also to measures not supported by X_η , but by its hereditary closure.

⁶This means that for all $\epsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the asymptotic upper density of the set $\bigcup_{b \in \mathcal{B}, b > k} b\mathbb{Z}$ does not exceed ϵ . It is known that the asymptotic density $d(\mathcal{F}_{\mathcal{B}}) = \lim_{N \rightarrow \infty} N^{-1} \text{card}(\mathcal{F}_{\mathcal{B}} \cap \{1, \dots, N\})$ exists when \mathcal{B} has light tails, and hence coincides with the so called logarithmic density of this set [6, Remark 2.29].

4 The map \mathcal{S}_H and the proof of Theorem A1

Throughout this section we assume that W is compact. Recall from (3) that, for each $\nu \in \mathcal{M}_W$, $\pi_*^H \nu$ is a measure on H , whose topological support is denoted by $\mathcal{S}_H(\nu)$. The set $\mathcal{S}_H(\nu) \subseteq W$ can be understood as the “minimal” window for ν in the following sense: Assume that $\nu \in \mathcal{M}_W$ satisfies $\nu \leq \nu_W(\hat{x})$ for some $\hat{x} \in \hat{X}$. Then the smallest compact set $W' \subseteq W$ such that $\nu \leq \nu_{W'}(\hat{x})$ is given by $W' = \mathcal{S}_H(\nu)$.

Denote $\widetilde{\mathcal{M}}_W := \{\nu \in \mathcal{M} : \nu \leq \nu_W(\hat{x}) \text{ for some } \hat{x} \in \hat{X}\}$. Then $\mathcal{M}_W = \overline{\nu_W(\hat{X})} \subseteq \widetilde{\mathcal{M}}_W$, because ν_W is upper semi-continuous. It is advantageous to view \mathcal{S}_H as a map from $\widetilde{\mathcal{M}}_W \setminus \{\underline{0}\}$ to \mathcal{K}_W , the space of all non-empty compact subsets of W , which is equipped with the topology generated by the Hausdorff distance.

Lemma 4.1. $\mathcal{S}_H : \widetilde{\mathcal{M}}_W \setminus \{\underline{0}\} \rightarrow \mathcal{K}_W$ is lower semicontinuous, i.e., for each closed $F \subseteq W$ the core of F , i.e. the set $\{\nu \in \widetilde{\mathcal{M}}_W \setminus \{\underline{0}\} : \mathcal{S}_H(\nu) \subseteq F\}$ is closed. In particular, \mathcal{S}_H is Borel measurable. The same holds for the restriction $\mathcal{S}_H|_{\mathcal{M}_W \setminus \{\underline{0}\}}$.

Proof. The above characterisation of lower semicontinuity is from [1, Prop. 1.4.4]. So let $\nu = \lim_{n \rightarrow \infty} \nu_n$ with $\mathcal{S}_H(\nu_n) \subseteq F$. Suppose for a contradiction that $\mathcal{S}_H(\nu)$ is not contained in F . Then, by closedness of F , it follows that there is $h \in \mathcal{S}_H(\nu) \setminus F$ such that $(\pi_*^H \nu)\{h\} = 1$. Hence there are $x \in G \times H$ and $\ell \in \mathcal{L}$ such that $(x + \ell)_H = h$ and $\nu\{x + \ell\} = 1$. As $\nu_n \rightarrow \nu$ vaguely, there are $x_n \in G \times H$ such that $x_n \rightarrow x$ and $\nu_n\{x_n + \ell\} = 1$ for all n . But then $(x_n + \ell)_H \in \mathcal{S}_H(\nu_n) \subseteq F$ for all n , and $(x_n + \ell)_H \rightarrow h$, so that $h \in F$, a contradiction. This proves the lower semicontinuity of \mathcal{S}_H , and its Borel measurability follows from [7, Cor. III.3]. As \mathcal{M}_W is a closed subset of $\widetilde{\mathcal{M}}_W$, these properties are inherited by the restriction $\mathcal{S}_H|_{\mathcal{M}_W \setminus \{\underline{0}\}}$. \square

Denote by $C_W \subseteq \hat{X}$ the set of continuity points of the map $\nu_W : \hat{X} \rightarrow \mathcal{M}_W$. It is a dense G_δ -set, see Proposition 8.3 below. An explicit characterization of this set is

$$C_W = \pi^{\hat{X}} \left(\bigcap_{\ell \in \mathcal{L}} (G \times (\partial W)^c) - \ell \right), \quad (6)$$

see Lemma 8.2 below or e.g. [15, Lem. 6.1].

Remark 4.2. $\mathcal{M}_{\min} = \overline{\nu_W(C_W)}$ is the unique minimal subset of \mathcal{M}_W by Lemma 8.4 below, see also [15, Thm. 1a]. Let $\mathcal{M}_{\min}^G := \pi_*^G(\mathcal{M}_{\min}) \subseteq \mathcal{M}_W^G$. Then \mathcal{M}_{\min}^G is the only minimal subsystem of \mathcal{M}_W^G . Even more, $\mathcal{M}_{\min}^G = \overline{\nu_W^G(C_W)}$. The \subseteq -inclusion follows, because $\overline{\nu_W^G(C_W)}$ is non-empty, closed and S -invariant. For the reverse inclusion observe that $\nu_W^G(C_W) = \pi_*^G(\nu_W(C_W)) \subseteq \pi_*^G(\mathcal{M}_{\min})$. If $\text{int}(W) = \emptyset$, then $\mathcal{M}_{\min} = \{\underline{0}\}$ is a singleton which consists only of the zero measure, see Lemma 8.4 or [15, Prop. 3.3].

Lemma 4.3. a) \mathcal{S}_H is S -invariant.

b) Let $\nu \in \widetilde{\mathcal{M}}_W$ and assume that $\nu' \in \overline{\mathcal{O}(\nu)} := \overline{\{S_g \nu : g \in G\}} \subseteq \widetilde{\mathcal{M}}_W$. Then $\mathcal{S}_H(\nu') \subseteq \mathcal{S}_H(\nu)$.

c) $\overline{\text{int}(W)} = \mathcal{S}_H(\nu)$ for all $\nu \in \mathcal{M}_{\min}$.

d) $\overline{\text{int}(W)} \subseteq \mathcal{S}_H(\nu)$ and $\text{int}(W) = \text{int}(\mathcal{S}_H(\nu))$ for all $\nu \in \mathcal{M}_W$.

Proof. a) $\mathcal{S}_H(S_g(\nu)) = \text{supp}(\pi_*^H(S_g(\nu))) = \text{supp}(\pi_*^H \nu) = \mathcal{S}_H(\nu)$ for all $g \in G$.

b) If $\nu = \underline{0}$ or $\nu' = \underline{0}$, the claim is trivial. Otherwise, the claim follows immediately from Lemma 4.1.

c) Assertion b) applies to any two $\nu, \nu' \in \mathcal{M}_{\min}$. Hence \mathcal{S}_H is constant on $\mathcal{M}_{\min} = \overline{\nu_W(C_W)}$. But for any continuity point \hat{x} of ν_W we have $\mathcal{S}_H(\nu_W(\hat{x})) = \overline{\text{int}(W)}$ by Remark 3.7.

d) Let $\nu \in \mathcal{M}_W$. As \mathcal{M}_{\min} is the unique minimal subset of \mathcal{M}_W , we have $\mathcal{M}_{\min} \subseteq \overline{\mathcal{O}(\nu)}$. Let $\nu' \in \mathcal{M}_{\min}$.

Then $\overline{\text{int}(W)} = \mathcal{S}_H(\nu')$ by part c), and $\mathcal{S}_H(\nu') \subseteq \mathcal{S}_H(\nu)$ by part a), because $\nu' \in \mathcal{M}_{\min} \subseteq \overline{\mathcal{O}(\nu)}$. Hence $\text{int}(W) \subseteq \overline{\text{int}(W)} \subseteq \mathcal{S}_H(\nu) \subseteq W$, in particular also $\text{int}(W) \subseteq \text{int}(\mathcal{S}_H(\nu)) \subseteq \text{int}(W)$. \square

Lemma 4.4. *Suppose $\nu, \nu' \in \widetilde{\mathcal{M}}_W$ and $\pi_*^G \nu = \pi_*^G \nu'$. Then there is $d \in H$ such that $\nu' = \sigma_d \nu$, where $(\sigma_d \nu)(A) := \nu(A - (0, d))$ for all Borel subsets A of $G \times H$. In particular, $d + \mathcal{S}_H(\nu') = \mathcal{S}_H(\nu)$ and $d + \text{int}(W) = \text{int}(W)$.*

Proof. (inspired by the proof of [15, Lem. 4.5])

If $\pi_*^G \nu = \pi_*^G \nu' = \underline{0}$, then also $\nu = \underline{0} = \nu'$, and the claim is obvious. Otherwise, by definition of $\widetilde{\mathcal{M}}_W$, there are $x, x' \in G \times H$ such that $\nu \leq \nu_W(x + \mathcal{L})$ and $\nu' \leq \nu_W(x' + \mathcal{L})$. Hence, $\pi_*^G \nu \leq \pi_*^G \nu_W(x + \mathcal{L})$ and $\pi_*^G \nu' \leq \pi_*^G \nu_W(x' + \mathcal{L})$, and as $\underline{0} \neq \pi_*^G \nu = \pi_*^G \nu'$, there is $\tilde{\ell} \in \mathcal{L}$ such that $x'_G = x_G + \tilde{\ell}_G$ by [15, Lem. 7.1d]. Let $\tilde{x} = x + \tilde{\ell}$. Then $\tilde{x}_G = x'_G$ and $\nu_W(x + \mathcal{L}) = \nu_W(\tilde{x} + \mathcal{L})$. Hence we can assume without loss of generality that $x_G = x'_G$. As $\pi^G|_{\mathcal{L}}$ is 1-1, we conclude that the following chain of equivalences holds for each $\ell \in \mathcal{L}$:

$$\nu\{x + \ell\} = 1 \quad \Leftrightarrow \quad \pi_*^G \nu\{x_G + \ell_G\} = 1 \quad \Leftrightarrow \quad \pi_*^G \nu'\{x'_G + \ell_G\} = 1 \quad \Leftrightarrow \quad \nu'\{x' + \ell\} = 1.$$

For $d := x'_H - x_H$ this can be rewritten as

$$\nu\{x' + \ell - (0, d)\} = 1 \quad \Leftrightarrow \quad \nu'\{x' + \ell\} = 1,$$

and as the measures ν and ν' are sums of unit point masses supported by the sets $x + \mathcal{L}$ and $x' + \mathcal{L}$, respectively, $\nu' = \sigma_d \nu$ follows at once. Hence $\text{supp}(\pi_*^H(\nu')) = \text{supp}(\pi_*^H(\nu)) - d$, so that $\mathcal{S}_H(\nu) = d + \mathcal{S}_H(\nu')$. Observing Lemma 4.3d, this implies $\text{int}(W) = d + \text{int}(W)$. \square

Proof of Theorem A1. a) As $\text{int}(W)$ is aperiodic, it is in particular nonempty. Hence (\mathcal{M}_W, S) is an almost 1-1 extension of its maximal equicontinuous factor (\hat{X}, \hat{T}) by [15, Thm. 1a]. As $\pi_*^G : (\mathcal{M}_W, S) \rightarrow (\mathcal{M}_W^G, S)$ is a continuous factor map between compact systems, all we have to show is that π_*^G is 1-1. So let $\nu, \nu' \in \mathcal{M}_W$ and suppose that $\pi_*^G(\nu) = \pi_*^G(\nu')$. Then $\nu' = \sigma_d \nu$ and $d + \mathcal{S}_H(\nu') = \mathcal{S}_H(\nu)$ for some $d \in H$ by Lemma 4.4. In particular $d + \text{int}(\mathcal{S}_H(\nu')) = \text{int}(\mathcal{S}_H(\nu))$, so that $d + \text{int}(W) = \text{int}(W)$ by Lemma 4.3d. By assumption, $\text{int}(W)$ is aperiodic. Therefore $d = 0$ and hence $\nu' = \sigma_0 \nu = \nu$.

b) As (\mathcal{M}_W^G, S) and (\mathcal{M}_W, S) are isomorphic by part a), all results for (\mathcal{M}_W, S) from [15] apply to (\mathcal{M}_W^G, S) as well. In particular, M contains the unique minimal invariant subset \mathcal{M}_{\min}^G of \mathcal{M}_W^G , and, just as (\mathcal{M}_W^G, S) itself, $(\mathcal{M}_{\min}^G, S)$ is an almost 1-1 extension of (\hat{X}, \hat{T}) with factor map $\Gamma|_{\mathcal{M}_{\min}^G}$ [15, Thm. 1a]. As $\mathcal{M}_{\min}^G \subseteq M \subseteq \mathcal{M}_W^G$, the claim of the theorem follows. \square

For later use we continue with some further lemmas highlighting properties of \mathcal{S}_H .

Lemma 4.5. *Let P, P' be ergodic S -invariant probability measures on \mathcal{M}_W .*

a) *There is a Haar regular subset $W_P \subseteq W$ of H such that $\mathcal{S}_H(\nu) = W_P$ for P -a.a. ν . It is empty if and only if $P(\{\underline{0}\}) = 1$.*

b) *If $P \circ (\pi_*^G)^{-1} = P' \circ (\pi_*^G)^{-1}$, then $W_P = W_{P'} + d$ for some $d \in H$.*

Proof. a) The claim is obvious for P satisfying $P(\{\underline{0}\}) = 1$, which is equivalent to $W_P = \emptyset$. Hence we assume without loss of generality that $P(\mathcal{M}_W \setminus \{\underline{0}\}) = 1$.

We need the following preparation. Fix a complete metric d on H that generates the topology, and also a countable dense subset $\{h_n : n \in \mathbb{N}\}$ of H . Define functions $\delta_n : \mathcal{M}_W \setminus \{\underline{0}\} \rightarrow \mathbb{R}$, $\delta_n(\nu) := d(h_n, \mathcal{S}_H(\nu))$. These functions inherit their Borel-measurability from \mathcal{S}_H [7, Thm. III.2 and III.9]. Note also that if $(\delta_n(\nu))_{n \in \mathbb{N}} = (\delta_n(\nu'))_{n \in \mathbb{N}}$, then $\mathcal{S}_H(\nu) = \mathcal{S}_H(\nu')$. Indeed, otherwise there is

$h \in \mathcal{S}_H(\nu) \setminus \mathcal{S}_H(\nu')$ (w.l.o.g.), and for a subsequence $(h_{n_k})_k$ converging to h one has $\lim_{k \rightarrow \infty} \delta_{n_k}(\nu) = d(h, \mathcal{S}_H(\nu)) = 0 < d(h, \mathcal{S}_H(\nu')) = \lim_{k \rightarrow \infty} \delta_{n_k}(\nu')$, a contradiction.

Now the \mathcal{S} -invariance of \mathcal{S}_H implies at once that all δ_n are \mathcal{S} -invariant. As P is ergodic, there are constants $(a_n)_{n \in \mathbb{N}}$ and a set $\mathcal{M}' \subseteq \mathcal{M}_W \setminus \{\underline{0}\}$ of full P -measure such that $\delta_n(\nu) = a_n$ for all $\nu \in \mathcal{M}'$ and all $n \in \mathbb{N}$. Hence $\mathcal{S}_H(\nu)$ is the same compact subset of H , call it W_P , for all $\nu \in \mathcal{M}'$. We have $W_P \neq \emptyset$, of course.

It remains to prove that W_P is Haar regular. Suppose for a contradiction that this is not the case. Then there are $h \in W_P$ and $r > 0$ such that $m_H(B_r(h) \cap W_P) = 0$, where $B_r(h) = \{h' \in H : d(h', h) < r\}$. In view of (3), $\pi^H(\text{supp}(\nu)) \cap B_r(h) \neq \emptyset$ for each $\nu \in \mathcal{M}'$. Using the torus map $\hat{\pi} : \mathcal{M}_W \setminus \{\underline{0}\} \rightarrow \hat{X}$, which was explained after Proposition 2.1, we infer $\pi^H(\text{supp}(\nu_w(\hat{\pi}(\nu)))) \cap B_r(h) \neq \emptyset$. Denote by $\pi(\nu)$ the unique representative of $\hat{\pi}(\nu)$ in the fundamental domain $X \subseteq G \times H$ of \hat{X} . It follows that

$$\mathcal{M}' \subseteq \bigcup_{\ell \in \mathcal{L}} \{\nu \in \mathcal{M}_W : \pi^H(\ell + \pi(\nu)) \in B_r(h) \cap W_P\} = \bigcup_{\ell \in \mathcal{L}} \pi^{-1}\left(X \cap \left((G \times (B_r(h) \cap W_P)) - \ell\right)\right).$$

In the remaining part of the proof we will show that $P(\mathcal{M}') = 0$, which is the desired contradiction. To that end recall that \mathcal{L} is countable and that $P \circ \hat{\pi}^{-1} = m_{\hat{X}}$, compare the proof of Proposition 2.2. Hence $P \circ \pi^{-1} = \text{dens}(\mathcal{L}) \cdot m_{G \times H}|_X$, compare Fact 2.2(2). Therefore it suffices to estimate

$$\begin{aligned} m_{G \times H}\left(X \cap \left((G \times (B_r(h) \cap W_P)) - \ell\right)\right) &\leq m_{G \times H}\left((G \times (B_r(h) \cap W_P)) - \ell\right) \\ &= m_{G \times H}(G \times (B_r(h) \cap W_P)), \end{aligned}$$

and to observe that the latter expression evaluates to 0, because $m_H(B_r(h) \cap W_P) = 0$.

b) Assume now that $P \circ (\pi_*^G)^{-1} = P' \circ (\pi_*^G)^{-1}$. In view of part a) of the lemma and of Lemma 4.4, there are $\nu, \nu' \in \mathcal{M}_W$ and $d \in H$ such that $W_P = \mathcal{S}_H(\nu) = d + \mathcal{S}_H(\nu') = d + W_{P'}$. \square

Lemma 4.6. *Suppose that W is aperiodic. Then $\pi_*^G : \mathcal{M}_W \rightarrow \mathcal{M}_W^G$ is 1-1 at $\nu \in \mathcal{M}_W$ whenever $\mathcal{S}_H(\nu) = W$.*

Proof. If $\mathcal{S}_H(\nu) = W$ and $\pi_*^G \nu = \pi_*^G \nu'$ for $\nu, \nu' \in \mathcal{M}_W$, then $\nu' = \sigma_d \nu$ and $W - d = \mathcal{S}_H(\nu) - d = \mathcal{S}_H(\nu') \subseteq W$ by Lemma 4.4. Let $W_n = W - n \cdot d$. Then $W_0 \supseteq W_1 \supseteq \dots$ is a nested sequence of compact sets. Suppose for a contradiction that there exists some $h_0 \in W_0 \setminus W_1$ and let $h_n := h_0 - n \cdot d \in W_n \subseteq W$ for $n \in \mathbb{N}$. Then, for $k > n$, $d(h_n, h_k) \geq d(h_n, W_k) \geq d(h_n, W_{n+1}) = d(h_0, W_1) > 0$, which is impossible, because W is compact. Hence $W = W_0 = W_1 = W - d$, so that $d = 0$ because W is aperiodic. \square

Lemma 4.7. *Suppose that $F \subseteq W$ is Haar regular and $K \subseteq \mathcal{M}_W$ is closed. Then $\pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) = F\}$ is a Borel subset of \mathcal{M}_W^G .*

Proof. The case $F = \emptyset$ is trivial. So we may assume that $F \neq \emptyset$. Denote by $V_1, V_2, \dots \subseteq H$ those elements of a base of the second countable space H , for which $F_n := F \setminus V_n$ is a proper subset of F . Then any compact proper subset F' of F is contained in some F_n . We thus can write

$$\pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) \subseteq F\} = \pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) = F\} \cup \left(\bigcup_{n=1}^{\infty} \pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) \subseteq F_n\} \right). \quad (7)$$

Next, let $\nu, \nu' \in \mathcal{M}_W$ with $\mathcal{S}_H(\nu) = F \neq \emptyset$, $\mathcal{S}_H(\nu') \subseteq F$ and $\pi_*^G(\nu') = \pi_*^G(\nu)$. In particular, $\nu, \nu' \neq \underline{0}$. Then $m_H(\mathcal{S}_H(\nu')) = m_H(\mathcal{S}_H(\nu)) = m_H(F)$ by Lemma 4.4. As F is Haar regular, this implies $\mathcal{S}_H(\nu') = F$. Therefore the union of the two sets on the rhs of Eqn. (7) is disjoint, and we have

$$\pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) = F\} = \pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) \subseteq F\} \setminus \bigcup_{n=1}^{\infty} \pi_*^G\{\nu \in K : \mathcal{S}_H(\nu) \subseteq F_n\}. \quad (8)$$

As all sets involved in the rhs of (8) are continuous images of sets which are compact by Lemma 4.1, the lhs of (8) is in particular Borel measurable. \square

5 Proofs of Theorems B1 and C1

We first prove both theorems for compact windows, and discuss the extension of Theorem B1 to windows which are compact modulo 0 in Subsection 8.2.

Fix any tempered van Hove sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of G , compare [15, Footnote 5]. We always have the upper bound

$$\bar{d}(\nu) := \limsup_{n \rightarrow \infty} \frac{\nu(A_n \times H)}{m_G(A_n)} \leq \text{dens}(\mathcal{L}) \cdot m_H(W)$$

on the upper density of any configuration $\nu \in \mathcal{M}_W$, see [15, Eqn. (14)]. We say that $\nu \in \mathcal{M}_W$ has *maximal density* if

$$d(\nu) := \lim_{n \rightarrow \infty} \frac{\nu(A_n \times H)}{m_G(A_n)} \text{ exists, and if } d(\nu) = \text{dens}(\mathcal{L}) \cdot m_H(W).$$

Recall from [21, Thm. 5a] that

$$Q_W \{ \nu \in \mathcal{M}_W : \nu \text{ has maximal density} \} = 1, \quad (9)$$

where $Q_W = m_{\hat{X}} \circ (\nu_W)^{-1}$ denotes the Mirsky measure on \mathcal{M}_W .

Lemma 5.1. $W_{\text{reg}} \subseteq \mathcal{S}_H(\nu) \subseteq W$ for each $\nu \in \mathcal{M}_W$ with maximal density.

Proof. We have $\mathcal{S}_H(\nu) \subseteq W$ for all $\nu \in \mathcal{M}_W$ by definition. If $m_H(W) = 0$, then $W_{\text{reg}} = \emptyset \subseteq \mathcal{S}_H(\nu)$. Assume now that $m_H(W) > 0$ and that ν has maximal density $d(\nu) = \text{dens}(\mathcal{L}) \cdot m_H(W) > 0$. In particular, $\nu \in \mathcal{M}_W \setminus \{0\}$. There is a unique $\hat{x} = x + \mathcal{L} \in \hat{X}$ such that $\nu \leq \nu_W(\hat{x})$ by [15, Lem. 5.4]. We thus get $\text{supp}(\nu) \subseteq (x + \mathcal{L}) \cap (G \times \mathcal{S}_H(\nu))$, which implies $\nu \leq \nu_{\mathcal{S}_H(\nu)}(\hat{x})$. Hence

$$\text{dens}(\mathcal{L}) \cdot m_H(W) = d(\nu) \leq \bar{d}(\nu_{\mathcal{S}_H(\nu)}(\hat{x})) \leq \text{dens}(\mathcal{L}) \cdot m_H(\mathcal{S}_H(\nu)),$$

which yields $m_H(W) = m_H(\mathcal{S}_H(\nu))$. As $\mathcal{S}_H(\nu)$ is a compact subset of W , this implies $W_{\text{reg}} \subseteq \mathcal{S}_H(\nu)$. \square

Corollary 5.2. $W_{Q_W} = W_{\text{reg}}$.

Proof. Observe that $W_{Q_W} = \mathcal{S}_H(\nu) \subseteq W$ for Q_W -a.a. ν by Lemma 4.5. Hence $W_{\text{reg}} \subseteq W_{Q_W}$ in view of (9) and Lemma 5.1. Haar regularity of W_{Q_W} , which holds by definition, implies $W_{\text{reg}} \supseteq W_{Q_W}$. Indeed, let $w \in W_{Q_W}$ and let U any neighborhood of w . Then $m_H(U \cap W) = m_H(U \cap W_{Q_W}) > 0$, because $m_H(W \setminus W_{Q_W}) \leq m_H(W \setminus W_{\text{reg}}) = 0$ and as W_{Q_W} is Haar regular. \square

Corollary 5.3. Denote by $\mathcal{M}'_W \subseteq \mathcal{M}_W$ the set of configurations of maximal density. If W is Haar aperiodic, then $\pi_*^G|_{\mathcal{M}'_W} : \mathcal{M}'_W \rightarrow \mathcal{M}_W^G$ is 1-1.

Proof. Note that $W_{\text{reg}} \subseteq \mathcal{S}_H(\nu) \subseteq W$ for any configuration $\nu \in \mathcal{M}'_W$ by Lemma 5.1, where $m_H(W \setminus W_{\text{reg}}) = 0$. Now assume that $\pi_*^G(\nu) = \pi_*^G(\nu')$ for $\nu, \nu' \in \mathcal{M}'_W$. Then $\nu' = \sigma_d \nu$ and $d + \mathcal{S}_H(\nu') = \mathcal{S}_H(\nu)$ for some $d \in H$ by Lemma 4.4, which implies

$$0 = m_H((d + \mathcal{S}_H(\nu')) \Delta \mathcal{S}_H(\nu)) = m_H((d + W) \Delta W).$$

As W is Haar aperiodic, we conclude $d = 0$ and thus $\nu = \nu'$. \square

Proof of Theorem B1. π_*^G is 1-1 at Q_W -a.a. $v \in M_W$ by Corollary 5.3 and Eqn. (9). Hence $\pi_*^G : (M_W, Q_W, S) \rightarrow (M_W^G, Q_W^G, S)$ is a measure theoretic isomorphism, and thus $\nu_W^G : (\hat{X}, m_{\hat{X}}, \hat{T}) \rightarrow (M_W^G, Q_W^G, S)$ is a measure theoretic isomorphism by Proposition 2.3. Here we use $m_H(W) > 0$, which follows from Haar aperiodicity of W . \square

We now turn to general S -invariant probability measures on M_W .

Corollary 5.4. Fix an ergodic S -invariant probability measure P on M_W and consider the Haar regular set $W_P \subseteq W$ from Lemma 4.5. Then $M_P := S_H^{-1}\{W_P\} \subseteq M_W$ has P -measure one. If W_P is aperiodic, then $\pi_*^G|_{M_P}$ is 1-1.

Proof. If $P = \delta_{\underline{0}}$, then $W_P = \emptyset$ and $M_P = \{\underline{0}\}$, and the claim is trivial. Otherwise we may assume that $P(M_W \setminus \{\underline{0}\}) = 1$. Then by Lemma 4.1, the set M_P is measurable. By Lemma 4.5 we have $P(M_P) = 1$. The injectivity of $\pi_*^G|_{M_P}$ follows from Lemma 4.4, where we use that W_P is aperiodic. \square

In order to infer results on M_W^G from M_W , we need to “lift” invariant probability measures from M_W^G to M_W . This is the content of the following proposition.

Proposition 5.5. Let P^G be an ergodic S -invariant probability measure on M_W^G , and denote by $\mathcal{P}(P^G)$ the family of all S -invariant probability measures P on M_W that project to P^G , i.e., for which $P \circ (\pi_*^G)^{-1} = P^G$. Then the following hold.

- a) $\mathcal{P}(P^G) \neq \emptyset$.
- b) Each $P \in \mathcal{P}(P^G)$ has an ergodic decomposition $P = \int P_e \circ \sigma_h^{-1} d\rho(h)$ for some compactly supported probability measure ρ on H and some ergodic S -invariant probability measure $P_e \in \mathcal{P}(P^G)$.
- c) If $P \in \mathcal{P}(P^G)$ is ergodic and if $W_P \subseteq W$ is aperiodic, then $\pi_*^G : M_W \rightarrow M_W^G$ is a measure theoretic isomorphism between (M_W, P, S) and (M_W^G, P^G, S) , and both systems are extensions of $(\hat{X}, m_{\hat{X}}, \hat{T})$.

Proof. a) Denote by Γ the set valued map (also called multifunction) from M_W^G to compact subsets of M_W defined by $\Gamma(v^G) = (\pi_*^G)^{-1}\{v^G\}$. It is measurable in the sense of [7, Thm. III.2], because $\Gamma^-(C) := \{v^G \in M_W^G : (\pi_*^G)^{-1}\{v^G\} \cap C \neq \emptyset\} = \pi_*^G(C)$ is compact and hence Borel measurable for any closed $C \subseteq M_W$. Hence, by the measurable selection theorem [7, Thm. III.6], there is a Borel measurable map $\psi_* : M_W^G \rightarrow M_W$ such that $\pi_*^G \circ \psi_* = \text{id}_{M_W^G}$. In particular, $P := P^G \circ (\psi_*)^{-1}$ is a well defined probability measure on M_W that projects to P^G . The measure P is not necessarily S -invariant, but a Krylov-Bogoliubov construction on P , see e.g. [8, Thm. 8.10], provides an S -invariant probability measure in $\mathcal{P}(P^G)$. The latter holds since also $(P \circ S_g^{-1})$ projects to P^G for every $g \in G$ as $(P \circ S_g^{-1}) \circ (\pi_*^G)^{-1} = P \circ (\pi_*^G \circ S_g)^{-1} = P \circ (S_g \circ \pi_*^G)^{-1} = P \circ (\pi_*^G)^{-1} \circ S_g^{-1} = P^G \circ S_g^{-1} = P^G$, and since the measure transport by π_*^G is continuous w.r.t. the weak topology.

b) For any $P \in \mathcal{P}(P^G)$, its ergodic decomposition [8, Thm. 8.20] can be written as

$$P = \int_{M_W} P_\mu dP(\mu), \quad (10)$$

where the probability measures P_μ on M_W are ergodic and where $\mu \mapsto P_\mu$ is Borel measurable. Then

$$P^G = P \circ (\pi_*^G)^{-1} = \int_{M_W} P_\mu \circ (\pi_*^G)^{-1} dP(\mu),$$

where all $P_\mu \circ (\pi_*^G)^{-1}$ are ergodic S -invariant measures on M_W^G . As P^G itself is ergodic, $P^G = P_\mu \circ (\pi_*^G)^{-1}$ for P -a.a. μ , so that $P_\mu \in \mathcal{P}(P^G)$ for P -a.a. μ .

Fix any measure P_e from the ergodic decomposition. Let P_μ be any other measure from this decomposition. P_e and P_μ can be disintegrated over P^G , namely there are systems $\{p_{\nu^G} : \nu^G \in \mathcal{M}_w^G\}$ and $\{p'_{\nu^G} : \nu^G \in \mathcal{M}_w^G\}$ of probability measures on \mathcal{M}_w , such that

$$P_e = \int_{\mathcal{M}_w^G} p_{\nu^G} dP^G(\nu^G) \quad \text{and} \quad P_\mu = \int_{\mathcal{M}_w^G} p'_{\nu^G} dP^G(\nu^G).$$

Hence, for P^G -a.a. $\nu^G \in \mathcal{M}_w^G$ and $(p_{\nu^G} \otimes p'_{\nu^G})$ -a.a. $(\nu, \nu') \in \mathcal{M}_w \times \mathcal{M}_w$, we have $\pi_*^G \nu = \nu^G = \pi_*^G \nu'$, so that $\nu' = \sigma_d \nu$ for some $d = d(\nu, \nu') \in H$. Since we may assume that ν and ν' are generic for P_e and P_μ , respectively, we can conclude $P_\mu = P_e \circ \sigma_d^{-1}$.

Consider the map $\kappa : H \rightarrow \mathcal{P}(P^G)$, $h \mapsto P_e \circ \sigma_h^{-1}$. It is continuous so that $\kappa(H)$ is a compact subset of the space of probability measures on \mathcal{M}_w . Observe that the set $\kappa(H)$ does not depend on the choice of a particular P_e in the definition of κ . As $P_\mu \in \kappa(H)$ for P -a.a. μ , we can rewrite the ergodic decomposition (10) as

$$P = \int_{\kappa(H)} \tilde{P} d\tilde{\rho}(\tilde{P}), \quad (11)$$

where $\tilde{\rho}$ is the distribution of the random measures P_μ under P .

The set valued map $\Gamma : P \mapsto \kappa^{-1}(\{P\})$ from $\mathcal{P}(P^G)$ to compact subsets of H is Borel measurable by the same arguments as in part a) of the proof. Hence, by the measurable selection theorem [7, Thm. III.6], there is a Borel measurable map $\kappa^\dagger : \kappa(H) \rightarrow H$ such that $\kappa \circ \kappa^\dagger = \text{id}_{\kappa(H)}$. In particular, $\rho := \tilde{\rho} \circ (\kappa^\dagger)^{-1}$ is a well defined probability measure on H , and

$$P = \int_{\kappa(H)} \kappa(\kappa^\dagger(\tilde{P})) d\tilde{\rho}(\tilde{P}) = \int_H \kappa(h) d\rho(h) = \int_H P_e \circ \sigma_h^{-1} d\rho(h). \quad (12)$$

c) If $P \in \mathcal{P}(P^G)$ is ergodic and if W_P is aperiodic, then $\pi_*^G : (\mathcal{M}_w, P, S) \rightarrow (\mathcal{M}_w^G, P^G, S)$ is a measure theoretic isomorphism in view of Corollary 5.4, and both systems are extensions of $(\hat{X}, m_{\hat{X}}, \hat{T})$ by Proposition 2.2. Here we use $m_H(W) \geq m_H(W_P) > 0$, as the aperiodic Haar regular set W_P is Haar aperiodic. \square

6 Periodic windows and quotient cut-and-project schemes

For a given a subset $A \subseteq H$, recall its period group $H_A = \{h \in H : h + A = A\}$. The set $A \subseteq H$ is (topologically) aperiodic, if $H_A = \{0\}$.

- Lemma 6.1.** a) $H_A \subseteq H_{\bar{A}} \cap H_{\text{int}(A)}$.
b) If A is closed, then also H_A is closed.
c) If $\text{int}(\bar{A}) = \text{int}(A)$ (e.g. if A is closed), then $H_{\text{int}(A)} = H_{\overline{\text{int}(A)}}$.
d) If A is compact and nonempty, then H_A is compact.

Proof. a) For each $h \in H$, the translation by h is a homeomorphism on H .

b) Let $h_n \in H_A$, $h = \lim_n h_n$. If $w \in A$, then $\pm h + w = \lim_n (\pm h_n + w) \in A$, because all $\pm h_n + w$ are in A and A is closed. This shows $\pm h + A \subseteq A$, i.e., $h + A = A$.

c) The assumption implies $\text{int}(\overline{\text{int}(A)}) = \text{int}(A)$. Indeed, $\text{int}(A) \subseteq \overline{\text{int}(A)}$ implies $\text{int}(A) \subseteq \text{int}(\overline{\text{int}(A)})$, and $\text{int}(A) \subseteq A$ implies $\text{int}(\overline{\text{int}(A)}) \subseteq \text{int}(\bar{A}) = \text{int}(A)$. Now the result follows from the implications

$$h + \text{int}(A) = \text{int}(A) \implies h + \overline{\text{int}(A)} = \overline{\text{int}(A)} \implies h + \text{int}(\overline{\text{int}(A)}) = \text{int}(\overline{\text{int}(A)}).$$

d) By definition we have $H_A + A = A$. As A is compact nonempty, H_A must be compact, too. \square

Remark 6.2. If the assumption $\text{int}(\overline{A}) = \text{int}(A)$ of part c) of the lemma is satisfied, then the topological boundary $\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \setminus \text{int}(\overline{A}) = \partial \overline{A}$ is nowhere dense. Both conditions are not equivalent, however, as is easily seen by looking at open dense sets A .

For a given cut-and-project scheme with window $W \subseteq H$, an important example is the period group H_W of the window. Some structural results for model sets rely on the assumption of an aperiodic window. Aperiodicity may however be assumed without loss of generality by passing to an associated quotient cut-and-project scheme, where the periods of the window have been factored out, compare [5, Section 9]. As this construction has not been fully described before, we present it here in some detail.

Let (G, H, \mathcal{L}) and $\hat{X} = (G \times H)/\mathcal{L}$ be as before, with quotient map $\pi^{\hat{X}} : G \times H \rightarrow \hat{X}$. Fix any compact subgroup $H_0 \subseteq H$ and consider $H' := H/H_0$ with factor map $\varphi : H \rightarrow H'$. Consider $\mathcal{H}_0 := \{0\} \times H_0 \subseteq G \times H$ and note that $\mathcal{L} \cap \mathcal{H}_0 = \{(0, 0)\}$ as π^G is 1-1 on \mathcal{L} . Observe next that $\iota : (G \times H)/\mathcal{H}_0 \rightarrow G \times H'$, $\iota((g, h) + \mathcal{H}_0) := (g, h + H_0)$, is a (rather trivial) isomorphism of topological groups. Denote by Φ the quotient map $\Phi : G \times H \rightarrow (G \times H)/\mathcal{H}_0$, and let $\mathcal{L}' := \iota(\Phi(\mathcal{L}))$.

Lemma 6.3. $\mathcal{L}' = \iota(\Phi(\mathcal{L}))$ is a discrete subgroup of $G \times H'$.

Proof. As $\iota \circ \Phi$ is a group homomorphism, \mathcal{L}' is a subgroup of $G \times H'$. We prove that \mathcal{L}' is discrete: Take a compact zero neighborhood $\mathcal{U} \subseteq G \times H$ such that $\mathcal{L} \cap \mathcal{U} = \{0\}$, which is possible since \mathcal{L} is discrete in $G \times H$. Now $\mathcal{L}' \cap \iota(\mathcal{U} + \mathcal{H}_0)$ contains 0 and is finite as $\mathcal{U} + \mathcal{H}_0 \subseteq G \times H$ is compact. Hence there is a zero neighborhood $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{L}' \cap \iota(\mathcal{V} + \mathcal{H}_0) = \{0\}$. \square

This lemma allows to consider the locally compact abelian quotient group $\widehat{X}' = (G \times H')/\mathcal{L}'$. Now (G, H', \mathcal{L}') is a cut-and-project scheme with associated torus \widehat{X}' , compare [5, Section 9], which we call a quotient cut-and-project scheme. For the convenience of the reader, we give a proof which is based on the following general facts about quotient groups.

Lemma 6.4. [11, Proposition III.20]

$(G \times H)/(\mathcal{L} + \mathcal{H}_0)$ is isomorphic (as a topological group) to each of the following groups:

- a) $\hat{X}/\pi^{\hat{X}}(\mathcal{L} + \mathcal{H}_0) = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_0)$
- b) $((G \times H)/\mathcal{H}_0)/\Phi(\mathcal{L} + \mathcal{H}_0) = ((G \times H)/\mathcal{H}_0)/\Phi(\mathcal{L})$

Corollary 6.5. In the above setting, (G, H', \mathcal{L}') is a cut-and-project scheme, in particular \mathcal{L}' is cocompact. The topological quotient group $\widehat{X}' = (G \times H')/\mathcal{L}'$ is isomorphic to $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_0)$.

Proof. Projection properties are inherited: Assume that $0 = \pi^G((\iota \circ \Phi)(\ell)) = \ell_G$ for some $\ell \in \mathcal{L}$. As π^G is 1-1 on \mathcal{L} , we infer $\ell = 0$, which implies $(\iota \circ \Phi)(\ell) = 0$. Note also that $\pi^{H'}(\mathcal{L}') = \varphi(\pi^H(\mathcal{L})) \supseteq \varphi(\overline{\pi^H(\mathcal{L})}) = \varphi(H) = H'$. Here we used continuity and surjectivity of the projection map, together with the assumption that $\pi^H(\mathcal{L})$ is dense in H .

As $G \times H' = \iota((G \times H)/\mathcal{H}_0)$ and $\mathcal{L}' = \iota(\Phi(\mathcal{L}))$, where ι is an isomorphism, we see that $(G \times H')/\mathcal{L}'$ is isomorphic to $((G \times H)/\mathcal{H}_0)/\Phi(\mathcal{L})$. Combining this with a) and b) of Lemma 6.4, we conclude that $(G \times H')/\mathcal{L}'$ is isomorphic to $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_0)$. As \mathcal{H}_0 is compact, $\pi^{\hat{X}}(\mathcal{H}_0)$ is compact, so that $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_0)$ is compact [11, Theorem III.11]. In particular, \mathcal{L}' is cocompact in $G \times H'$. \square

Remark 6.6. The factor map $\iota \circ \Phi : G \times H \rightarrow G \times H'$ carries over to a factor map $\iota\widehat{\Phi} : \hat{X} \rightarrow \widehat{X}'$, because $\iota(\Phi(x + \mathcal{L})) = \iota(\Phi(x)) + \mathcal{L}'$. It pushes the Haar measure $m_{\hat{X}}$ to the Haar measure $m_{\widehat{X}'}$ on \widehat{X}' .

If $W \subseteq H$ is a window in (G, H, \mathcal{L}) and H_0 is a closed subgroup of H , we can consider the quotient cut-and-project scheme (G, H', \mathcal{L}') with window $W' := \varphi(W)$, because W' inherits the basic topological properties from W :

Lemma 6.7. a) W' is compact.

b) If W is topologically regular, then so is W' .

c) If W is Haar regular, then so is W' .

Proof. a) Recall that $\varphi : H \rightarrow H'$ is continuous and open [11, Theorem III.10]. In particular, $W' = \varphi(W)$ is compact.

b) If W is topologically regular, i.e. if $W = \overline{\text{int}(W)}$, then

$$W' = \varphi(\overline{\text{int}(W)}) \subseteq \overline{\varphi(\text{int}(W))} \subseteq \overline{\text{int}(\varphi(W))} = \overline{\text{int}(W')},$$

where we used continuity of φ for the first inclusion and openness for the second one.

c) Suppose now that W is Haar regular, and let $U \subseteq H'$ be open with $m_{H'}(U \cap W') = 0$. We must show that $U \cap W' = \emptyset$. By definition of the quotient topology, $\varphi^{-1}(U)$ is open in H . As $\varphi^{-1}(U) \cap W \subseteq \varphi^{-1}(U \cap W')$, we have $m_H(\varphi^{-1}(U) \cap W) \leq m_H(\varphi^{-1}(U \cap W')) = m_{H'}(U \cap W') = 0$, so that $\varphi^{-1}(U) \cap W = \emptyset$, because W is Haar regular. This implies $U \cap W' = U \cap \varphi(W) = \emptyset$. \square

An important example is $H_0 := H_W$, the period group of a window, because W' is aperiodic in this case. We will study the relations between the two cut-and-project schemes (G, H, \mathcal{L}) and (G', H', \mathcal{L}') with associated windows W and W' , respectively. For clarity we write \mathcal{H}_W instead of \mathcal{H}_0 and add the index W also to the quotient maps. Recall that the quotient map $\iota \circ \Phi_W : G \times H \rightarrow G \times H'$ is given by $x \mapsto (\iota \circ \Phi_W)(x) = \iota(x + \mathcal{H}_W) = (x_G, x_H + H_W)$.

Lemma 6.8. The window $W' = \varphi_W(W)$ is an aperiodic subset of H' .

Proof. Suppose that $W' + h' = W'$ for some $h' = \varphi_W(h) \in H'$. Then $\varphi_W(W + h) = \varphi_W(W)$, in particular $W + h + H_W = W + H_W$. As $W + H_W = W$, this shows that $h \in H_W$, so that $h' = \varphi_W(h)$ is the neutral element of H' . \square

Lemma 6.9. Let $x \in G \times H$ and $x' \in G \times H'$ be such that $x' = (\iota \circ \Phi)(x)$, and let $W' = \varphi_W(W)$.

a) The sets $(x + \mathcal{L}) \cap (G \times W)$ and $(x' + \mathcal{L}') \cap (G \times W')$ are in 1-1 correspondence via the quotient map $\iota \circ \Phi$. In particular we have $\pi^G((x + \mathcal{L}) \cap (G \times W)) = \pi^G((x' + \mathcal{L}') \cap (G \times W'))$.

b) For every $h \in H_W$ we have $\nu_W(x + (0, h) + \mathcal{L}) = \sigma_h \nu_W(x + \mathcal{L})$, where $\sigma_h \nu(A) = \nu(A - (0, h))$.

c) We have $\nu_{W'}(x' + \mathcal{L}') = \sum_{y \in (x + \mathcal{L}) \cap (G \times W)} \delta_{\iota(y + \mathcal{H}_W)}$, which implies $\pi_*^G(\nu_{W'}(x' + \mathcal{L}')) = \pi_*^G(\nu_W(x + \mathcal{L}))$, i.e. $\nu_{W'}^G(x' + \mathcal{L}') = \nu_W^G(x + \mathcal{L})$.

Proof. a) To show injectivity of the quotient map, assume that $(\iota \circ \Phi)(x + \ell_1) = (\iota \circ \Phi)(x + \ell_2)$, that is $x + \ell_1 + \mathcal{H}_W = x + \ell_2 + \mathcal{H}_W$. We thus can conclude $\ell_1 - \ell_2 \in \{0\} \times H_W$. Hence $\ell_{1,G} - \ell_{2,G} = 0$, and as π^G is 1-1 on \mathcal{L} , we infer $\ell_{1,H} - \ell_{2,H} = 0$. Hence $\ell_1 = \ell_2$, and we get $x + \ell_1 = x + \ell_2$. To show that the quotient map is onto, assume without loss of generality that W is nonempty. Take arbitrary $y' \in (x' + \mathcal{L}') \cap (G \times W')$. Then $y' = \iota(x + \ell + \mathcal{H}_W)$ for some $\ell \in \mathcal{L}$, which implies $x_H + \ell_H + H_W \in W' = \{w + H_W : w \in W\}$. This implies $x_H + \ell_H \in W + H_W = W$, which means that $y := x + \ell \in (x + \mathcal{L}) \cap (G \times W)$. The remaining statement is now obvious, since we have $\pi^G(x) = \pi^G((\iota \circ \Phi)(x))$.

b) Let $h \in H_W$ be given. A direct calculation yields

$$\nu_W(x + (0, h) + \mathcal{L}) = \sum_{y \in (x + (0, h) + \mathcal{L}) \cap (G \times W)} \delta_y = \sum_{y \in (x + \mathcal{L}) \cap (G \times W)} \delta_{y + (0, h)} = \sigma_h \nu_W(x + \mathcal{L}).$$

c) A direct calculation yields

$$\nu_{W'}(x' + \mathcal{L}') = \sum_{y' \in (x' + \mathcal{L}') \cap (G \times W')} \delta_{y'} = \sum_{\iota(y + \mathcal{H}_W) \in (x' + \mathcal{L}') \cap (G \times W')} \delta_{\iota(y + \mathcal{H}_W)} = \sum_{y \in (x + \mathcal{L}) \cap (G \times W)} \delta_{\iota(y + \mathcal{H}_W)},$$

where we use a) for the third equality. The remaining statement is now obvious. \square

Proposition 6.10. *Let $W' = \varphi_W(W)$.*

a) $\mathcal{M}_W^G = \pi_*^G(\mathcal{M}_W) = \pi_*^G(\mathcal{M}_{W'}^G) = \mathcal{M}_{W'}^G$.

b) $\mathcal{Q}_W^G = m_{\hat{X}} \circ (\nu_W^G)^{-1} = m_{\hat{X}'} \circ (\nu_{W'}^G)^{-1} = \mathcal{Q}_{W'}^G$.

Proof. a) Note the following chain of equivalences:

$$\begin{aligned} \nu^G \in \pi_*^G(\nu_W(\hat{X})) &\Leftrightarrow \exists x \in G \times H : \nu^G = \pi_*^G(\nu_W(x + \mathcal{L})) \\ &\Leftrightarrow \exists x' \in G \times H' : \nu^G = \pi_*^G(\nu_{W'}(x' + \mathcal{L}')) \\ &\Leftrightarrow \nu^G \in \pi_*^G(\nu_{W'}(\hat{X}')), \end{aligned}$$

where we used Lemma 6.9 c). This means that $\pi_*^G(\nu_W(\hat{X})) = \pi_*^G(\nu_{W'}(\hat{X}'))$. Now a) of the proposition follows from continuity of π_*^G and compactness of $\mathcal{M}_W = \nu_W(\hat{X}) \subseteq \mathcal{M}$ and $\mathcal{M}_{W'}^G = \nu_{W'}(\hat{X}') \subseteq \mathcal{M}'$, the space of locally finite measures on the Borel subsets of $G \times H'$.

b) In view of Lemma 6.9c, $(\nu_{W'}^G \circ \iota\hat{\Phi})(x + \mathcal{L}) = \nu_{W'}^G(\iota(\Phi(x)) + \mathcal{L}') = \nu_W^G(x + \mathcal{L})$ for all $x \in G \times H$. Hence, observing Remark 6.6, $m_{\hat{X}'} \circ (\nu_{W'}^G)^{-1} = m_{\hat{X}} \circ \iota\hat{\Phi}^{-1} \circ (\nu_W^G)^{-1} = m_{\hat{X}} \circ (\nu_W^G)^{-1}$. \square

7 Proofs of Theorems A2, B2 and C2

In this section, W is again a compact window. We begin with a technical lemma that will be used at several places.

Lemma 7.1. *Let $\nu, \nu' \in \mathcal{M}_W$ and $W_0 \subseteq W$ be such that $\pi_*^G(\nu') = \pi_*^G(\nu)$ and $\mathcal{S}_H(\nu) = \mathcal{S}_H(\nu') = W_0$. Then $\gamma(\nu') - \gamma(\nu) \in \pi^{\hat{X}}(\mathcal{H}_{W_0})$.*

Proof. As $\pi_*^G(\nu) = \pi_*^G(\nu')$, Lemma 4.4 implies that $\nu' = \sigma_d \nu$ for some $d \in H_{W_0}$. Recall that $\gamma(\nu)$ equals $x + \mathcal{L}$ for any point $x \in G \times H$ with $\nu\{x\} = 1$. But $\nu\{x\} = \nu'\{x + (0, d)\}$, so $\gamma(\nu') = x + (0, d) + \mathcal{L}$. Hence $\gamma(\nu') - \gamma(\nu) = (0, d) + \mathcal{L} \in \pi^{\hat{X}}(\mathcal{H}_{W_0})$. \square

Suppose that $\text{int}(W) \neq \emptyset$. Denote by γ the factor map from \mathcal{M}_W onto its maximal equicontinuous factor \hat{X} ⁷, and by ρ the factor map from \hat{X} onto $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_{\text{int}(W)})$. We define a factor map Γ from \mathcal{M}_W^G to $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_{\text{int}(W)})$ as follows: for $\nu^G \in \mathcal{M}_W^G$ pick any $\nu \in (\pi_*^G)^{-1}\{\nu^G\}$ and let $\Gamma(\nu^G) = \rho(\gamma(\nu))$.

Lemma 7.2. *Suppose that $\text{int}(W) \neq \emptyset$.*

a) Γ is well defined.

b) Γ is continuous and commutes with the dynamics.

c) If $H_{\text{int}(W)} = H_W$ and $\Gamma(\nu^G) = \rho(\hat{x})$ for some $\nu^G \in \mathcal{M}_W^G$ and $\hat{x} \in C_W$, then $\nu^G = \pi_*^G(\nu_W(\hat{x}))$.

d) If $H_{\text{int}(W)} = H_W$, then Γ is almost 1-1.

Proof. a) Suppose that $\pi_*^G(\nu) = \pi_*^G(\nu')$. Then $\gamma(\nu') - \gamma(\nu) \in \pi^{\hat{X}}(\mathcal{H}_{\text{int}(W)})$ by Lemma 7.1, so that $\rho(\gamma(\nu')) = \rho(\gamma(\nu) + \pi^{\hat{X}}(\mathcal{H}_{\text{int}(W)})) = \rho(\gamma(\nu))$. This shows that Γ is well defined.

b) Let $D \subseteq \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{\text{int}(W)})$ be closed. Then $E := \gamma^{-1}(\rho^{-1}(D))$ is closed in \mathcal{M}_W , and

$$\nu^G \in \Gamma^{-1}(D) \Leftrightarrow \exists \nu \in E : \pi_*^G(\nu) = \nu^G \Leftrightarrow \nu^G \in \pi_*^G(E). \quad (13)$$

As E is also compact, this shows that $\Gamma^{-1}(D) = \pi_*^G(E)$ is closed. Hence Γ is continuous. We show that it commutes with the dynamics: Let $\nu^G \in \mathcal{M}_W^G$, $g \in G$, and denote $D := \{\Gamma(\nu^G)\}$ and $E := \gamma^{-1}(\rho^{-1}(D))$. Then $\nu^G \in \Gamma^{-1}(D) = \pi_*^G(E)$ by (13), and

$$S_g \circ \gamma^{-1} \circ \rho^{-1} = (\rho \circ \gamma \circ S_{-g})^{-1} = (\hat{T}_{-g} \circ \rho \circ \gamma)^{-1} = \rho^{-1} \circ \gamma^{-1} \circ \hat{T}_g.$$

⁷This is the map $\pi_*^{\hat{X}} \circ (\pi_*^{G \times H})^{-1}$ from [15, Thm. 1a].

Hence $S_g \nu^G \in S_g(\pi_*^G(E)) = \pi_*^G(S_g(\gamma^{-1}(\rho^{-1}(D)))) = \pi_*^G(\gamma^{-1}(\rho^{-1}(\hat{T}_g(D)))) = \Gamma^{-1}(\hat{T}_g(D))$, where we used again (13) for the last identity. Therefore, $\Gamma(S_g \nu^G) \in \hat{T}_g(D) = \{\hat{T}_g(\Gamma(\nu^G))\}$.

c) Let $h \in H_W$. Then $\hat{x} \in C_W$ if and only if $\hat{x} + (0, h) \in C_W$. This follows immediately from

$$\nu_W(\hat{x} + (0, h)) = \sigma_h(\nu_W(\hat{x})), \quad (14)$$

see Lemma 6.9b).

Suppose now that $\hat{x} \in C_W$ and $\Gamma(\nu^G) = \rho(\hat{x})$ for some $\nu^G \in \mathcal{M}_W^G$. There is $\nu \in \mathcal{M}_W$ such that $\pi_*^G(\nu) = \nu^G$ and $\rho(\gamma(\nu)) = \rho(\hat{x})$. Hence $\gamma(\nu) \in \hat{x} + \mathcal{H}_{\text{int}(W)} = \hat{x} + \mathcal{H}_W$, i.e., there is $h \in H_W$ such that $\gamma(\nu) = \hat{x} + (0, h) \in C_W$. This implies $\nu = \nu_W(\hat{x} + (0, h))$, see [15, Prop. 3.3b]. Hence

$$\nu^G = \pi_*^G(\nu) = \pi_*^G(\nu_W(\hat{x} + (0, h))) = \pi_*^G(\nu_W(\hat{x}) + (0, h)) = \pi_*^G(\nu_W(\hat{x})),$$

where we used (14) for the third identity.

d) In view of assertion c), $\Gamma^{-1}\{\rho(\hat{x})\}$ is a singleton for each $\rho(\hat{x}) \in \rho(C_W)$. For countable acting groups G it is well known that this implies that Γ is almost 1-1. For uncountable groups we could not locate such a statement in the literature. So we provide a proof for the convenience of the reader: In view of c) we only need to show that $\rho(C_W)$ is a dense G_δ -set in $\hat{X}/\pi^\times(\mathcal{H}_{\text{int}(W)})$: C_W is a dense G_δ -set in \hat{X} by [15, Prop. 3.3c]. Hence denseness of $\rho(C_W)$ follows as ρ is continuous and onto. In (14) we argued that C_W - and hence also $\hat{X} \setminus C_W$ - are invariant under translations by elements from the subgroup H_W . Hence $\rho(\hat{X} \setminus C_W) = (\hat{X}/\pi^\times(\mathcal{H}_{\text{int}(W)})) \setminus \rho(C_W)$. Indeed, suppose $\rho(\hat{x}) = \rho(\hat{y})$ for some $\hat{x} \in C_W$ and $\hat{y} \in \hat{X} \setminus C_W$. Then $\hat{y} \in \hat{x} + \mathcal{H}_{\text{int}(W)} = \hat{x} + \mathcal{H}_W \subseteq C_W$, a contradiction. To show that $\rho(C_W)$ is a G_δ -set, it now suffices to show that $\rho(\hat{X} \setminus C_W)$ is an F_σ -set. But this is obvious since $\hat{X} \setminus C_W$ is an F_σ -set by Eqn. (6) and hence a countable union of compact sets, and since ρ is continuous. \square

Proof of Theorem A2. a) Let M be any non-empty, closed S -invariant subset of \mathcal{M}_W^G . (\hat{X}', \hat{T}') is a factor of (M, S) by Lemma 7.2. We prove that it is the maximal equicontinuous factor of (M, S) .

Let $W_0 := \overline{\text{int}(W)}$ and $H_0 := H_{W_0} = H_{\text{int}(W)}$ (see Lemma 6.1 for the second identity). Then $W' = \varphi_{W_0}(W_0)$ is an aperiodic subset of $H' = H/H_{W_0}$ by Lemma 6.8. As W_0 is topologically regular, also W' is topologically regular (Lemma 6.7). Hence also $\text{int}(W')$ is aperiodic.

Let $\hat{X}' = \hat{X}/\pi^\times(\mathcal{H}_{\text{int}(W)})$ with induced G -action \hat{T}' , and recall from Corollary 6.5 that \hat{X}' is isomorphic to $(G \times H')/\mathcal{L}'$. Theorem A1 implies that $(\mathcal{M}_{W'}^G, S)$ is an almost 1-1 extension of (\hat{X}', \hat{T}') . As $\mathcal{M}_{W'}^G = \pi_*^G(\mathcal{M}_{W'}) = \pi_*^G(\mathcal{M}_{W_0}) = \mathcal{M}_{W_0}^G$ by Proposition 6.10 (applied to W_0 instead of W), also $(\mathcal{M}_{W_0}^G, S)$ is an almost 1-1 extension of (\hat{X}', \hat{T}') . It follows that also the unique minimal subsystem $(\overline{\nu_{W_0}^G(C_{W_0})}, S)$ of $(\mathcal{M}_{W_0}^G, S)$ is an almost 1-1 extension of (\hat{X}', \hat{T}') .

Note next that $C_W \subseteq C_{W_0}$ and $\nu_{W_0}^G|_{C_W} = \nu_W^G|_{C_W}$, because $\partial W_0 \subseteq \partial W$. As C_W is dense in \hat{X} (and as C_{W_0} is the set of continuity points of $\nu_{W_0}^G$), this implies $\overline{\nu_{W_0}^G(C_{W_0})} = \overline{\nu_W^G(C_W)} = \mathcal{M}_{\min}^G$. Hence the minimal system $(\mathcal{M}_{\min}^G, S)$ is an almost 1-1 extension of (\hat{X}', \hat{T}') , so that (\hat{X}', \hat{T}') is the maximal equicontinuous factor of $(\mathcal{M}_{\min}^G, S)$.

Suppose now that (\tilde{X}, \tilde{T}) is an equicontinuous factor of (M, S) and observe that $\mathcal{M}_{\min}^G \subseteq M$. Then the restriction of the factor map to \mathcal{M}_{\min}^G defines a factor map from $(\mathcal{M}_{\min}^G, S)$ to (\tilde{X}, \tilde{T}) . It follows that (\tilde{X}, \tilde{T}) is a factor of (\hat{X}', \hat{T}') . As this holds for any equicontinuous factor (\tilde{X}, \tilde{T}) of (M, S) , the system (\hat{X}', \hat{T}') is in fact the maximal equicontinuous factor of (M, S) .

b) This is Lemma 7.2d. \square

Remark 7.3. A proof of Theorem A2 only for the case when $H_{\text{int}(W)} = H_W$ is much simpler: In that case $H' = H/H_W$, and $W' = \varphi_W(W)$ is aperiodic by Lemma 6.8. As $\mathcal{M}_W^G = \mathcal{M}_{W'}^G$ by Proposition 6.10a, all assertions of Theorem A2 follow from Theorem A1 applied to the cut-and-project scheme (G, H', \mathcal{L}') with window W' .

Proof of Theorem B2. Assume first that the window W is Haar regular. Then $W = W_{\text{reg}} = W_{Q_W}$ by Corollary 5.2. Let $W' = \varphi_W(W)$. This set is Haar regular by Lemma 6.7 and aperiodic by Lemma 6.8. Hence W' is also Haar aperiodic, see Remark 3.12. As $(\mathcal{M}_W^G, Q_W^G, S) = (\mathcal{M}_{W'}^G, Q_{W'}^G, S)$ by Proposition 6.10, the claim of the theorem follows now from Theorem B1.

In the general case, note that (\mathcal{M}_W, Q_W, S) is measure theoretically isomorphic to (\mathcal{M}, Q_W, S) . As the present theorem applies to the Haar regularized window W_{reg} , we must only show that $Q_W = m_{\hat{X}} \circ (\nu_W)^{-1}$ equals $Q_{W_{\text{reg}}} = m_{\hat{X}} \circ (\nu_{W_{\text{reg}}})^{-1}$ on \mathcal{M} . But this follows from the observation that

$$\{\hat{x} \in \hat{X} : \nu_W(\hat{x}) \neq \nu_{W_{\text{reg}}}(\hat{x})\} \subseteq \pi^{\hat{X}} \left(\bigcup_{\ell \in \mathcal{L}} ((G \times (W \setminus W_{\text{reg}})) - \ell) \right),$$

and this is a set of $m_{\hat{X}}$ -measure zero, because \mathcal{L} is countable and $m_H(W \setminus W_{\text{reg}}) = 0$. \square

Proof of Theorem C2. Let P^G be an ergodic S -invariant probability measure on $\mathcal{M}_W^G \setminus \{0\}$. Take any ergodic S -invariant probability measure P on $\mathcal{M}_W \setminus \{0\}$ satisfying $P^G = P \circ (\pi_*^G)^{-1}$. In particular we have $W_P \neq \emptyset$. Let $\hat{X}' = \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{W_P})$ with induced G -action \widehat{T}' and Haar measure $m_{\hat{X}'}$. Denote

$$\mathcal{A}_P := \{\nu \in \mathcal{M}_W : S_H(\nu) = W_P\},$$

where $W_P \subseteq W$ is the Haar regular set from Lemma 4.5, for which $S_H(\nu) = W_P$ for P -a.a. ν . Hence $P(\mathcal{A}_P) = 1$, and \mathcal{A}_P is Borel measurable and S -invariant, because S_H is (Lemmas 4.1 and 4.3a). Then also $\pi_*^G(\mathcal{A}_P) \subseteq \mathcal{M}_W^G \setminus \{0\}$ is S -invariant, it is Borel measurable by Lemma 4.7, and $P^G(\pi_*^G(\mathcal{A}_P)) = P((\pi_*^G)^{-1}(\pi_*^G(\mathcal{A}_P))) \geq P(\mathcal{A}_P) = 1$.

Denote by γ the factor map from $\mathcal{M}_W \setminus \{0\}$ onto \hat{X} ⁸, and by ρ the factor map from \hat{X} onto $\hat{X}/\pi^{\hat{X}}(\mathcal{H}_{W_P})$. We define now a map

$$\Gamma : \pi_*^G(\mathcal{A}_P) \rightarrow \hat{X}/\pi^{\hat{X}}(\mathcal{H}_{W_P}), \quad \{\Gamma(\nu^G)\} := (\rho \circ \gamma)((\pi_*^G)^{-1}\{\nu^G\} \cap \mathcal{A}_P).$$

In order to see that Γ is well defined, observe first the cardinality is at least 1, because $\nu^G \in \pi_*^G(\mathcal{A}_P)$. On the other hand, if $\nu, \nu' \in (\pi_*^G)^{-1}\{\nu^G\} \cap \mathcal{A}_P$, then $\gamma(\nu') - \gamma(\nu) \in \pi^{\hat{X}}(\mathcal{H}_{W_P})$ by Lemma 7.1, i.e. $(\rho \circ \gamma)(\nu') = (\rho \circ \gamma)(\nu)$.

Next observe that Γ commutes with the dynamics: For each $g \in G$,

$$\begin{aligned} \{\Gamma(S_g \nu^G)\} &= (\rho \circ \gamma)((\pi_*^G)^{-1}(S_g \{\nu^G\}) \cap \mathcal{A}_P) = (\rho \circ \gamma)(S_g((\pi_*^G)^{-1}\{\nu^G\}) \cap S_g(\mathcal{A}_P)) \\ &= \widehat{T}'_g((\rho \circ \gamma)((\pi_*^G)^{-1}\{\nu^G\} \cap \mathcal{A}_P)) = \widehat{T}'_g\{\Gamma(\nu^G)\} \\ &= \{\widehat{T}'_g(\Gamma(\nu^G))\} \end{aligned}$$

It remains to show that $\pi_*^G(\mathcal{A}_P) \cap \Gamma^{-1}(K')$ is Borel measurable for each closed subset K' of \hat{X}' . Then it follows that Γ is Borel measurable and $(\hat{X}', m_{\hat{X}'}, \widehat{T}')$ is a measure theoretic factor of $(\mathcal{M}_W^G, P^G, S)$ as claimed in Theorem C2.

⁸This is the map $\pi_*^{\hat{X}} \circ (\pi_*^{G \times H})^{-1}$ from [15, Thm. 1a].

So let $K := (\rho \circ \gamma)^{-1}(K') \subseteq \mathcal{M}_w$. Then K is closed, and

$$K \cap \mathcal{A}_P = (\pi_*^G)^{-1}(\pi_*^G(K \cap \mathcal{A}_P)) \cap \mathcal{A}_P. \quad (15)$$

The \subseteq -inclusion is trivial. To see the reverse inclusion, let $v \in \mathcal{A}_P$ and assume that there exists $v' \in K \cap \mathcal{A}_P$ such that $\pi_*^G(v) = \pi_*^G(v')$. Then $\gamma(v) - \gamma(v') \in \pi^\chi(\mathcal{H}_{W_P})$ by Lemma 7.1, so that $(\rho \circ \gamma)(v) \in (\rho \circ \gamma)(v') + \rho(\pi^\chi(\mathcal{H}_{W_P})) = (\rho \circ \gamma)(v') \in (\rho \circ \gamma)(K) \subseteq K'$, i.e. $v \in (\rho \circ \gamma)^{-1}(K') = K$.

Now let $v^G \in \pi_*^G(\mathcal{A}_P)$. Then

$$\begin{aligned} \Gamma(v^G) \in K' &\Leftrightarrow \{\Gamma(v^G)\} \subseteq K' \\ &\Leftrightarrow (\rho \circ \gamma)\left((\pi_*^G)^{-1}\{v^G\} \cap \mathcal{A}_P\right) \subseteq K' \\ &\Leftrightarrow (\pi_*^G)^{-1}\{v^G\} \cap \mathcal{A}_P \subseteq (\rho \circ \gamma)^{-1}(K') = K \\ &\Leftrightarrow (\pi_*^G)^{-1}\{v^G\} \cap \mathcal{A}_P \subseteq K \cap \mathcal{A}_P \\ &\Leftrightarrow v^G \in \pi_*^G(K \cap \mathcal{A}_P). \end{aligned}$$

The last equivalence is seen as follows:

„ \Rightarrow ”: As $v^G \in \pi_*^G(\mathcal{A}_P)$, there exists some $v \in (\pi_*^G)^{-1}\{v^G\} \cap \mathcal{A}_P \subseteq F \cap \mathcal{A}_P$, so that $v^G = \pi_*^G(v) \in \pi_*^G(F \cap \mathcal{A}_P)$.

„ \Leftarrow ”: Let $v \in (\pi_*^G)^{-1}\{v^G\} \cap \mathcal{A}_P$. Then $\pi_*^G(v) = v^G \in \pi_*^G(F \cap \mathcal{A}_P)$, so that in view of (15),

$$v \in (\pi_*^G)^{-1}(\pi_*^G(F \cap \mathcal{A}_P)) \cap \mathcal{A}_P = F \cap \mathcal{A}_P.$$

Hence $\pi_*^G(\mathcal{A}_P) \cap \Gamma^{-1}(K') = \pi_*^G(K \cap \mathcal{A}_P)$, and this set is Borel measurable by Lemma 4.7. \square

8 Relatively compact windows

For suitable relatively compact windows $W \subseteq H$, dynamical properties are the same as in the compact case. This has already been observed in [15, Rem. 3.16].

8.1 Topological results

For some topological results, we assume that the boundary ∂W is nowhere dense. This condition characterises denseness of the set C_W of continuity points of the map $v_W : \mathcal{M}_W \rightarrow \hat{X}$, see Proposition 8.3. As any compact window has nowhere dense boundary, this condition generalises the compact case. The next lemma shows that this condition also generalises topological regularity.

Lemma 8.1. *For each $W \subseteq H$, ∂W is nowhere dense if and only if $\overline{\text{int}(\overline{\text{int}(W)})} = \overline{\text{int}(W)}$.*

Proof. As $\partial W = \overline{W} \cap \overline{W^c}$, we have $\text{int}(\partial W) = \text{int}(\overline{W}) \cap \text{int}(\overline{W^c})$. Hence ∂W is nowhere dense if and only if $\text{int}(\overline{W}) \subseteq (\text{int}(\overline{W^c}))^c = \overline{\text{int}(W)}$. But this is obviously equivalent to $\overline{\text{int}(\overline{\text{int}(W)})} = \overline{\text{int}(W)}$. \square

The next lemma generalises slightly Lemma 6.1 in [15].

Lemma 8.2. *Let $W \subseteq H$ be relatively compact. Then the set $C_W \subseteq \hat{X}$ of continuity points of the map $v_W : \hat{X} \rightarrow \mathcal{M}_W$ is given by*

$$C_W = \pi^\chi \left(\bigcap_{\ell \in \mathcal{L}} ((G \times (\partial W)^c) - \ell) \right).$$

Proof. This is seen by reinspecting the proof of [15, Lem. 6.1]. If $x_H \notin \bigcap_{\ell \in \mathcal{L}} ((\partial W)^c - \ell_H)$, then there is $\ell \in \mathcal{L}$ such that $x_H + \ell_H \in \partial W$. In the case $x_H + \ell_H \in W$ we take $x_H^n \in (W)^c - \ell_H$ such that $x_H^n \rightarrow x_H$ as $n \rightarrow \infty$ and let $x^n = (x_G, x_H^n)$. Then, for each sufficiently small open neighbourhood U of $x + \ell$ in $G \times H$ and sufficiently large n we have $\nu_w(x^n + \mathcal{L})(U) = 1_W(x_H^n + \ell_H) \cdot \delta_{x_n + \ell}(U) = 0$ while $\nu_w(x + \mathcal{L})(U) = 1_W(x_H + \ell_H) \cdot \delta_{x + \ell}(U) = 1$ so that $\nu_w(x^n + \mathcal{L}) \not\rightarrow \nu_w(x + \mathcal{L})$. In particular, $(x + \mathcal{L}) \notin C_w$. If $x_H + \ell_H \in (W)^c$, we may argue analogously with a sequence $x_H^n \in W - \ell_H$ such that $x_H^n \rightarrow x_H$ as $n \rightarrow \infty$.

Conversely, let $x + \mathcal{L}, x^n + \mathcal{L} \in \hat{X}$, $\lim_{n \rightarrow \infty} (x^n + \mathcal{L}) = (x + \mathcal{L})$ and take a sufficiently small open neighbourhood U of $x + \ell$ in $G \times H$. Assume that $x_H \in \bigcap_{\ell \in \mathcal{L}} ((\partial W)^c - \ell_H)$, i.e., $(x + \ell)_H \notin \partial W$ for all $\ell \in \mathcal{L}$. Now consider $\ell \in \mathcal{L}$ such that $x_H + \ell_H \in W$. Then by assumption $x_H + \ell_H \in \text{int}(W)$, which implies $1 = \nu_w(x + \mathcal{L})(U) = \nu_w(x^n + \mathcal{L})(U)$ for sufficiently large n . If $\ell \in \mathcal{L}$ such that $x_H + \ell_H \notin W$, then by assumption $x_H + \ell_H \in (\overline{W})^c$, which implies $0 = \nu_w(x + \mathcal{L})(U) = \nu_w(x^n + \mathcal{L})(U)$ for sufficiently large n . This implies $\lim_{n \rightarrow \infty} \nu_w(x^n + \mathcal{L}) = \nu_w(x + \mathcal{L})$ and hence $x + \mathcal{L} \in C_w$. \square

The following proposition collects some further properties of the set C_w .

Proposition 8.3. *Let $W \subseteq H$ be relatively compact. Then the set $C_w \subseteq \hat{X}$ of continuity points of the map $\nu_w : \hat{X} \rightarrow \mathcal{M}_w$ is a G_δ -set. The set C_w is dense in \hat{X} if and only if ∂W is nowhere dense in H . Otherwise $C_w = \emptyset$.*

*Proof.*⁹ Let $A := \bigcap_{\ell \in \mathcal{L}} ((G \times (\partial W)^c) - \ell)$ and $B := (G \times H) \setminus A = \bigcup_{\ell \in \mathcal{L}} G \times (\partial W - \ell_H)$, and recall that $\hat{X} = (G \times H)/\mathcal{L}$. As both sets, A and B , are invariant under translations by elements from the lattice \mathcal{L} , $\pi^\times(B) = \hat{X} \setminus \pi^\times(A)$. Indeed, suppose $\pi^\times(a) = \pi^\times(b)$ for some $a \in A$ and $b \in B$. Then $b \in a + \mathcal{L} \subseteq A$, a contradiction. To show that $\pi^\times(A)$ is a G_δ -set, it suffices to show that $\pi^\times(B)$ is a F_σ -set. To that end recall that G is σ -compact, i.e. there are compact $K_1, K_2, \dots \subseteq G$ such that $G = \bigcup_{j \in \mathbb{N}} K_j$. Hence $\pi^\times(B) = \bigcup_{\ell \in \mathcal{L}} \bigcup_{j \in \mathbb{N}} \pi^\times(K_j \times (\partial W - \ell_H))$ is a countable union of compact sets, hence F_σ .

Now assume that ∂W is nowhere dense. Then $A_H := \bigcap_{\ell \in \mathcal{L}} ((\partial W)^c - \ell_H)$ is dense in H as ∂W is nowhere dense and as H is a Baire space. This readily implies that $A = (\pi^H)^{-1}(A_H) = G \times A_H$ is dense in $G \times H$. Now denseness of $\pi^\times(A) = C_w$ in \hat{X} follows as π^\times is continuous and onto.

Conversely, assume that ∂W is not nowhere dense. Then there is some nonempty open set $O \subseteq \partial W$, which implies $A_H^c \supseteq \bigcup_{\ell \in \mathcal{L}} (O - \ell_H) = H$ because of the denseness of $\pi^H(\mathcal{L})$ in H . Thus $A_H = \emptyset$ and $C_w = \pi^\times((\pi^H)^{-1}(A_H)) = \emptyset$, in particular C_w is not dense. \square

Let us denote by $\mathcal{G}\nu_w := \{(\hat{x}, \nu_w(\hat{x})) : \hat{x} \in \hat{X}\} \subseteq \hat{X} \times \mathcal{M}_w$ the graph of the map ν_w , and by $\mathcal{G}\mathcal{M}_w$ its closure in the vague topology. Likewise, denote by $\mathcal{G}(\nu_w|_{C_w})$ the restriction of $\mathcal{G}\nu_w$ to its continuity points. We have the following general result on minimal subsets of $\mathcal{G}\mathcal{M}_w, \mathcal{M}_w$ and \mathcal{M}_w^G .

Lemma 8.4. *Let $W \subseteq H$ be relatively compact. Then*

- The set $\overline{\mathcal{G}(\nu_w|_{C_w})}$ is the unique minimal subset of $\mathcal{G}\mathcal{M}_w$.*
- The set $\overline{\nu_w(C_w)}$ is the unique minimal subset of \mathcal{M}_w .*
- The set $\overline{\nu_w^G(C_w)}$ is the unique minimal subset of \mathcal{M}_w^G .*
- If $\text{int}(W) = \emptyset$, then $\overline{\nu_w(C_w)}$ and $\overline{\nu_w^G(C_w)}$ are singletons consisting of the zero measure only.*

Proof. This can be seen by re-inspecting the proofs in [15], but we give a simple direct argument for the ease of the reader.

- Let $\emptyset \neq A \subseteq \mathcal{G}\mathcal{M}_w$ be any closed invariant set. Then $\emptyset \neq \pi^\times(A) \subseteq \hat{X}$ is closed invariant. Hence

⁹For compact windows this was claimed in [15, Prop. 3.3c]. As the proof of that proposition contains a mistake, we provide a full proof for the more general case treated here. (Indeed, the argument given in footnote \dagger of the proof of that proposition is wrong, because, with the sets U_ℓ and X defined there, it is not true that $\pi^\times(X \cap U_\ell) = \pi^\times(U_\ell)$.)

$\pi_*^\delta(A) = \hat{X} \supseteq C_W$, since (\hat{X}, \hat{T}) is minimal. As $(\pi_*^\delta)^{-1}\{\hat{x}\} \cap \mathcal{GM}_W = \{(\hat{x}, \nu_W(\hat{x}))\}$ for each $\hat{x} \in C_W$, this implies $A \supseteq \mathcal{G}(\nu_W|_{C_W})$, which means $A \supseteq \overline{\mathcal{G}(\nu_W|_{C_W})} =: A_{\min}$.

b) Let $\emptyset \neq B \subseteq \mathcal{M}_W$ be any closed invariant set. Then $\emptyset \neq (\pi_*^{G \times H})^{-1}(B) \subseteq \mathcal{GM}_W$ is closed invariant. By the previous result, we infer $(\pi_*^{G \times H})^{-1}(B) \supseteq A_{\min}$. Hence $B \supseteq \pi_*^{G \times H}(A_{\min}) \supseteq \nu_W(C_W)$, so that $\overline{\nu_W(C_W)} \subseteq \overline{B} = B$.

c) This follows using the same argument as in b).

d) If $\text{int}(W) = \emptyset$, then $(\partial W)^c \subseteq (W)^c$. Hence $\hat{x} = x + \mathcal{L} \in C_W$ implies $x_H \in \bigcap_{\ell \in \mathcal{L}} ((\partial W)^c - \ell_H) \subseteq \bigcap_{\ell \in \mathcal{L}} ((W)^c - \ell_H)$. But the latter condition means $(x + \mathcal{L}) \cap (G \times W) = \emptyset$. Hence $C_W \subseteq Z_W$, and the claim follows. \square

Consider now the window \overline{W} , as well. We infer from [15, Lem. 5.4] that, for each $\nu \in \mathcal{M}_W \setminus \{\underline{0}\}$, there is a unique $\hat{\nu}(\nu) \in \hat{X}$ such that $\text{supp}(\nu) \subseteq \text{supp}(\nu_{\overline{W}}(\hat{\nu}(\nu)))$. Thus the map $\hat{\pi} : \mathcal{M}_W \setminus \{\underline{0}\} \rightarrow \hat{X}$ is still well-defined and continuous in our more general setting, and it satisfies $\hat{\pi} = \pi_*^\delta \circ (\pi_*^{G \times H})^{-1}$. We have the following version of [15, Thm. 1a].

Proposition 8.5. *Let $W \subseteq H$ be relatively compact and such that ∂W nowhere dense in H . Assume that $\text{int}(\overline{W})$ is nonempty (being equivalent to $\text{int}(W)$ nonempty, in this case). Then*

a) $\pi_*^\delta : (\mathcal{GM}_W, S) \rightarrow (\hat{X}, \hat{T})$ is a topological almost 1-1-extension of its maximal equicontinuous factor.

b) $\hat{\pi} : (\mathcal{M}_W, S) \rightarrow (\hat{X}, \hat{T})$ is a topological almost 1-1-extension of its maximal equicontinuous factor.

Proof. a) We can argue as in the proof of [15, Prop. 3.5c]. The assumption ∂W nowhere dense guarantees that C_W is a dense G_δ -set by Proposition 8.3.

b) This follows from a) by noting that the statements and proofs of [15, Prop. 3.5b] and [15, Prop. 3.3e] still apply to the present situation. \square

If ∂W is nowhere dense, then \mathcal{M}_W and $\mathcal{M}_{\overline{W}}$ have the same unique minimal subset, and a similar result holds for the G -projections.

Lemma 8.6. (See also [15, Cor. 1b and Remark 3.16]) *Let $W \subseteq H$ be relatively compact and such that ∂W nowhere dense in H . Then $\overline{\nu_W(C_W)} = \overline{\nu_{\overline{W}}(C_{\overline{W}})}$ and $\overline{\nu_W^G(C_W)} = \overline{\nu_{\overline{W}}^G(C_{\overline{W}})}$.*

Proof. As $\partial \overline{W} \subseteq \partial W$, we have $C_W \subseteq C_{\overline{W}}$ and $\nu_W|_{C_W} = \nu_{\overline{W}}|_{C_W}$, because $1_{\overline{W}}(h) = 1_W(h)$ for all $h \in H \setminus \partial W$. Hence $\overline{\nu_W(C_W)} = \overline{\nu_{\overline{W}}(C_W)} \subseteq \overline{\nu_{\overline{W}}(C_{\overline{W}})}$. On the other hand, as C_W is dense in \hat{X} and as $\nu_{\overline{W}}$ is continuous on $C_{\overline{W}}$, we have $\overline{\nu_{\overline{W}}(C_{\overline{W}})} \subseteq \overline{\nu_W(C_W)}$. This proves the first identity. The second identity follows at once, because $\nu_W^G = \pi_*^G \circ \nu_W$ with a continuous π_*^G . \square

Now we are ready to state and prove the following extensions of Theorems A1 and A2.

Theorem A1'. *Let $W \subseteq H$ be relatively compact and ∂W be nowhere dense. Assume that $\text{int}(\overline{W})$ is aperiodic (so in particular non-empty).*

a) *The topological dynamical systems (\mathcal{M}_W, S) and (\mathcal{M}_W^G, S) are isomorphic, and both are almost 1-1 extensions of their maximal equicontinuous factor (\hat{X}, \hat{T}) .*

b) *Denote by $\Gamma : \mathcal{M}_W^G \rightarrow \hat{X}$ the factor map from a). If M is a non-empty, closed S -invariant subset of \mathcal{M}_W^G , then (M, S) is an almost 1-1 extension of its maximal equicontinuous factor (\hat{X}, \hat{T}) with factor map $\Gamma|_M$.*

Sketch of proof of Theorem A1'. In Section 4, the space $\widetilde{\mathcal{M}_{\overline{W}}}$ was defined such that $\mathcal{M}_W \subseteq \widetilde{\mathcal{M}_{\overline{W}}}$. Hence Lemma 4.1 applies also to $\mathcal{S}_H|_{\mathcal{M}_W \setminus \{\underline{0}\}}$. In view of Lemma 8.6, Lemma 4.3c) and d) and Lemma 4.4 remain valid. Keeping in mind the above results, one readily checks that the proof of Theorem A1 also applies under the above assumptions. \square

Theorem A2'. Let $W \subseteq H$ be relatively compact and ∂W be nowhere dense. Assume that $\text{int}(\overline{W}) \neq \emptyset$. Let $\widehat{X}' = \widehat{X}/\pi^{\hat{x}}(\mathcal{H}_{\text{int}(\overline{W})})$ with induced G -action \widehat{T}' , and let M be any non-empty, closed S -invariant subset of \mathcal{M}_W^G (thus including the case $M = \mathcal{M}_W^G$).

a) $(\widehat{X}', \widehat{T}')$ is the maximal equicontinuous factor of the topological dynamical system (M, S) .

b) If $H_{\text{int}(\overline{W})} = H_{\overline{W}}$, then (M, S) is an almost 1-1 extension of $(\widehat{X}', \widehat{T}')$.

Sketch of proof of Theorem A2'. Here we note that $H_{\text{int}(\overline{W})}$ is closed due to Lemma 6.1b) and c). In particular, the factor group $\widehat{X}/\pi^{\hat{x}}(\mathcal{H}_{\text{int}(\overline{W})})$ is Hausdorff. This ensures that all arguments in the proof of Theorem A2 for compact windows directly apply to the present situation. \square

8.2 Measure theoretic results

For measure theoretic results, let us assume that $W \subseteq H$ is relatively compact and measurable. In that situation, the map $\nu_W : \widehat{X} \rightarrow \mathcal{M}_W \subseteq \mathcal{M}$ is still measurable such that the Mirsky measure is well defined, compare [15, Rem. 3.16]. In fact Propositions 2.1, 2.2 and 2.3 continue to hold. In particular, (\mathcal{M}_W, Q_W, S) is a measure theoretic factor of $(\widehat{X}, m_{\widehat{X}}, \widehat{T})$, and thus the same holds for $(\mathcal{M}_W^G, Q_W^G, S)$. Hence both systems have pure point dynamical spectrum.

Whereas the statement of Proposition 2.1 is obvious from measurability of ν_W , we give proofs of the other propositions for the convenience of the reader.

Proof of Proposition 2.2. Note that $P \circ \hat{\pi}^{-1}$ is a probability measure on \widehat{X} by assumption on P . As $P \circ \hat{\pi}^{-1}$ is \widehat{T} -invariant and the \widehat{T} -action is minimal, it thus equals $m_{\widehat{X}}$, compare Fact 2.2 (3). In particular $0 = P \circ (\hat{\pi})^{-1}(Z_W) = m_{\widehat{X}}(Z_W)$, where $Z_W \subseteq \widehat{X}$ is the set of points $\hat{x} \in \widehat{X}$ such that $\nu_W(\hat{x}) = \underline{0}$. But this implies $m_H(W) > 0$ by [15, Prop. 3.6b], which continues to hold in the non-compact setting. The assertion of the proposition now follows as the desired factor map [8, Def. 2.7] is provided by $\hat{\pi}$. \square

Proof of Proposition 2.3. As $m_H(W) > 0$, we have $m_{\widehat{X}}(Z_W) = 0$ by [15, Prop. 3.6b], which continues to hold in the non-compact setting. Hence we have $Q_W(\mathcal{M}_W \setminus \{\underline{0}\}) = m_{\widehat{X}} \circ (\nu_W)^{-1}(\mathcal{M}_W \setminus \{\underline{0}\}) = m_{\widehat{X}}(\widehat{X} \setminus Z_W) = 1 - m_{\widehat{X}}(Z_W) = 1$. We thus can combine the statements of Proposition 2.2 and Proposition 2.1 to get the result. \square

Remark 8.7. (Generic configurations)

Assume that some configuration $\nu_W^G(\hat{x})$ has maximal density in the sense that $d(\nu_W^G(\hat{x})) = \text{dens}(\mathcal{L}) \cdot m_H(\overline{W})$ along some given tempered van Hove sequence $(A_n)_n$, compare also Section 5. In that case $\nu_W^G(\hat{x})$ is generic for the Mirsky measure Q_W^G on \mathcal{M}^G .

To see this, note that for any $\hat{x} \in \widehat{X}$ we have $0 \leq \nu_W^G(\hat{x}) \leq \nu_W^G(\hat{x})$, so in particular the density of $\nu_W^G(\hat{x})$ along $(A_n)_n$ is bounded by that of $\nu_W^G(\hat{x})$ and hence by $\text{dens}(\mathcal{L}) \cdot m_H(\overline{W})$. If $\nu_W^G(\hat{x})$ achieves this maximal density, then the density of $\nu_W^G(\hat{x}) - \nu_W^G(\hat{x})$ is clearly zero. Now consider for $\nu_W^G(\hat{x}) \in \mathcal{M}_W^G$ its empirical measures

$$Q_{W,\hat{x}}^{G,n} := \frac{1}{m_G(A_n)} \int_{A_n} \delta_{S_g \nu_W^G(\hat{x})} dm_G(g),$$

and likewise for $\nu_W^G(\hat{x}) \in \mathcal{M}_W^G$ its empirical measures $Q_{W,\hat{x}}^{G,n}$. The previous reasoning shows that the empirical measures $Q_{W,\hat{x}}^{G,n}$ are asymptotically equivalent to the measures $Q_{W,\hat{x}}^{G,n}$ in the sense that both sequences do have the same weak limit points, and [15, Thm. 5] implies that the measures $Q_{W,\hat{x}}^{G,n}$ converge weakly to Q_W^G . It follows that statistical properties of $\nu_W^G(\hat{x})$ and $\nu_W^G(\hat{x})$, like pattern frequencies and especially their autocorrelation coefficients, coincide for such \hat{x} , and that they are determined by the Mirsky measure Q_W^G , to which Theorems B1 and B2 apply.

Note, however, that for $m_{\hat{X}}$ -a.a \hat{x} the configuration $\nu_w(\hat{x})$ has density $\text{dens}(\mathcal{L}) \cdot m_H(W)$, see [21, Thm. 1]. Thus the above reasoning applies to Mirsky-typical configurations only, if $m_H(W) = m_H(\overline{W})$.

Remark 8.8. (Autocorrelation measure, diffraction spectrum and generic configurations)

Denote $\widetilde{\mathcal{M}}_W^G := \{\nu \in \mathcal{M}^G : \nu \leq \nu_W^G(\hat{x}) \text{ for some } \hat{x} \in \hat{X}\}$. For any probability measure P^G on $\widetilde{\mathcal{M}}_W^G$, the so-called autocorrelation measure γ_{P^G} is a positive definite measure on G that is naturally associated to P^G . In particular, γ_{P^G} is Fourier transformable. We recall its definition from [4, Prop. 6], [16, Sec. 2.3] and [17, Lemma 4.1]. Take any $\psi \in C_c(G)$ such that $m_G(\psi) = 1$ and define the measure γ_{P^G} via its associated linear functional $\gamma_{P^G} : C_c(G) \rightarrow \mathbb{C}$ by requiring

$$\gamma_{P^G}(\varphi) := \int_{\widetilde{\mathcal{M}}_W^G} \left(\int_G \int_G \varphi(s-t) \psi(t) d\nu(s) d\nu(t) \right) dP^G(\nu)$$

for every $\varphi \in C_c(G)$. The measure γ_{P^G} is independent of the choice of ψ , and it is a rather direct consequence of the above definition that the map $\gamma : \mathcal{M}(\widetilde{\mathcal{M}}_W^G) \rightarrow \mathcal{M}(G)$, defined by $P^G \mapsto \gamma_{P^G}$, is continuous with respect to the vague topologies. For $P^G := Q_W^G$ the Mirsky measure on $\widetilde{\mathcal{M}}_W^G$, its Fourier transform $\gamma_{Q_W^G}$ is a pure point measure as $(\widetilde{\mathcal{M}}_W^G, Q_W^G, S)$ has pure point dynamical spectrum [4, Thm. 7]. One says that $(\widetilde{\mathcal{M}}_W^G, Q_W^G, S)$ has pure point diffraction spectrum.

Now fix any configuration $\nu^G \in \widetilde{\mathcal{M}}_W^G$ and consider its empirical measures $Q_{\nu^G}^n$ on $\widetilde{\mathcal{M}}_W^G$, given as

$$Q_{\nu^G}^n := \frac{1}{m_G(A_n)} \int_{A_n} \delta_{S_g \nu^G} dm_G(g),$$

along some fixed tempered van Hove sequence $(A_n)_n$ in G . As $\widetilde{\mathcal{M}}_W^G$ is compact, we can assume that the associated sequence $(\gamma_{Q_{\nu^G}^n})_n$ of empirical autocorrelations converges to a limit γ_{ν^G} , possibly after passing to some subsequence of $(A_n)_n$. For sufficiently large n , the Fourier transform of γ_{ν^G} describes the outcome of a diffraction experiment on a physical realisation of ν^G restricted to A_n . A standard tedious calculation which we omit yields the explicit expression $\gamma_{\nu^G} = \sum_{\ell \in \mathcal{L}} \eta(\ell) \delta_{\ell_G}$, where

$$\eta(\ell) = \lim_{n \rightarrow \infty} \frac{1}{m_G(A_n)} \text{card}(\text{supp}(\nu^G) \cap (\ell_G + \text{supp}(\nu^G)) \cap A_n) \geq 0$$

are the autocorrelation coefficients of γ_{ν^G} . In particular, if ν^G is generic for the Mirsky measure Q_W^G on $\widetilde{\mathcal{M}}_W^G$, i.e., if $(Q_{\nu^G}^n)_n$ converges to Q_W^G , then by continuity of γ we have $\gamma_{\nu^G} = \gamma_{Q_W^G}$. By the above argument, this implies that ν^G is pure point diffractive. In particular, from [15, Eqn. (18)] we obtain explicit expressions for the autocorrelation coefficients, namely $\eta(\ell) = \text{dens}(\mathcal{L}) \cdot m_H(W \cap (W + \ell_H))$ for all ℓ such that $\eta(\ell) \neq 0$.

The above result alternatively follows by combining Theorem 5 and Proposition 6 in [4]. That there is a full Q_W^G -measure set of configurations $\nu_W^G(\hat{x}) \in (\nu_W^G)^{-1}(\hat{X}) \subseteq \mathcal{M}_W^G$ such that $\gamma_{\nu_W^G(\hat{x})} = \gamma_{Q_W^G}$ on any given tempered van Hove sequence $(A_n)_n$ has been shown by Moody [21, Cor. 1], by constructing a full $m_{\hat{X}}$ -measure set of pure point diffractive $\nu_W^G(\hat{x})$ via repeated applications of Birkhoff's ergodic theorem.

A sufficient criterion for pure point diffractiveness has been observed in [3, 15]: The configuration $\nu_W^G(\hat{x})$ is pure point diffractive if $\nu_W^G(\hat{x})$ has maximal density $d(\nu_W^G(\hat{x})) = \text{dens}(\mathcal{L}) \cdot m_H(\overline{W})$. Indeed, in that case $\nu_W^G(\hat{x})$ is generic for the Mirsky measure Q_W^G on $\widetilde{\mathcal{M}}_W^G$ by Remark 8.7 above. Note however that such $\nu_W^G(\hat{x})$ is typically not generic for the Mirsky measure Q_W^G on $\widetilde{\mathcal{M}}_W^G$.

For general relatively compact measurable windows W , it might be difficult to give an isomorphism between $(\mathcal{M}_W^G, \mathcal{Q}_W^G, S)$ and an explicit group rotation. But the previous results for compact windows, i.e., Theorems B1 and B2, continue to hold for windows W that are compact modulo 0. Consider a window W that is compact modulo 0 and its Haar regularization $W_{reg} = \text{supp}((m_H)|_W)$. Then the Mirsky measures \mathcal{Q}_W and $\mathcal{Q}_{W_{reg}}$ coincide on \mathcal{M} , because

$$\{\hat{x} \in \hat{X} : \nu_{W_{reg}}(\hat{x}) \neq \nu_W(\hat{x})\} \subseteq \pi^{\mathbb{X}} \left(\bigcup_{\ell \in \mathcal{L}} ((G \times (W_{reg} \Delta W)) - \ell) \right),$$

which is a set of $m_{\hat{X}}$ -measure zero as \mathcal{L} is countable. This implies that $(\mathcal{M}_W, \mathcal{Q}_W, S)$ is measure theoretically isomorphic to $(\mathcal{M}_{W_{reg}}, \mathcal{Q}_{W_{reg}}, S)$. As the Haar periods of W_{reg} coincide with those of W , Theorems B1 and B2, which apply to W_{reg} , continue to hold for W .

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